Fluctuation theory of Lévy Processes and Applications

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Part 0: Fluctuation Theories

Lévy Processes

Defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t; t \geq 0\}$ be a Lévy process, i.e.

1. The paths are almost surely right continuous with left limits.
2. For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.
3. For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

Wiener-Hopf factorization and Duality

- Let X be a Lévy process with a Laplace exponent:

$$\psi(s) := \log \mathbb{E}\left[e^{sX_1}\right] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{sz} - 1 - sz1_{\{|z|<1\}})\nu(dz),$$

  such that $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge z^2)\nu(dz) < \infty$.
- The Wiener-Hopf factorization states that

$$q/(q - \psi(s)) = \varphi_q^+(s)\varphi_q^-(s).$$

  with the Wiener-Hopf factors:

$$\varphi_q^-(s) := \mathbb{E}\left[\exp(s\underline{X}_{e_q})\right] \quad \text{and} \quad \varphi_q^+(s) := \mathbb{E}\left[\exp(s\overline{X}_{e_q})\right],$$

  where e_q is an independent exponential r.v. with parameter q.
- Duality lemma: the pairs $(\overline{X}_t, \overline{X}_t - X_t)$ and $(X_t - \underline{X}_t, -\underline{X}_t)$ have the same law.

Resolvent measures for Lévy processes

- Resolvents:

$$\Theta^{(q)}(x,dy) = \int_0^\infty e^{-qt}\mathbb{P}_x\{X_t \in dy\}\,dt = \frac{1}{q}\mathbb{P}_x\{X_{e_q} \in dy\}.$$

- We have

$$\Theta^{(q)}(x,dy) = \frac{1}{q}\mathbb{P}\left\{(X_{e_q} - \underline{X}_{e_q}) + \underline{X}_{e_q} \in dy - x\right\} = \frac{1}{q}\int_{[x-y,\infty)} \mathbb{P}\left\{-\underline{X}_{e_q} \in dz\right\}\mathbb{P}\left\{X_{e_q} - \underline{X}_{e_q} \in dy - x + z\right\}.$$

- By duality, $X_{e_q} - \underline{X}_{e_q} \sim \overline{X}_{e_q}$ and hence

$$\Theta^{(q)}(x,dy) = \frac{1}{q}\int_{[x-y,\infty)} \mathbb{P}\left\{-\underline{X}_{e_q} \in dz\right\}\mathbb{P}\left\{\overline{X}_{e_q} \in dy - x + z\right\}.$$

Spectrally Negative Lévy Processes

- Let X be a spectrally negative Lévy process with a Laplace exponent:

$$\psi(s) := \log \mathbb{E}\left[e^{sX_1}\right] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} (e^{sz} - 1 - sz1_{\{-1<z<0\}})\nu(dz),$$

  such that $\int_{(-\infty,0)}(1 \wedge z^2)\nu(dz) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} z\,\nu(dz) < \infty$.
- We exclude the case X is a subordinator.

Wiener-Hopf Factorization revisited

1. Recall the Wiener-Hopf factorization:

$$\mathrm{q}/(\mathrm{q}-\psi(\mathrm{s})) = \varphi_{\mathrm{q}}^{+}(\mathrm{s})\varphi_{\mathrm{q}}^{-}(\mathrm{s}).$$

   with the Wiener-Hopf factors:

$$\varphi_{\mathrm{q}}^{-}(\mathrm{s}) := \mathbb{E}\left[\exp(\mathrm{s}\underline{\mathrm{X}}_{\mathrm{e_q}})\right] \quad \text{and} \quad \varphi_{\mathrm{q}}^{+}(\mathrm{s}) := \mathbb{E}\left[\exp(\mathrm{s}\overline{\mathrm{X}}_{\mathrm{e_q}})\right],$$

2. For the spectrally negative case, $\overline{\mathrm{X}}_{\mathrm{e_q}}$ is exponentially distributed with parameter

$$\Phi(\mathrm{q}) := \sup\{\lambda \geq 0 : \psi(\lambda) = \mathrm{q}\}.$$

3. Hence, we know both $\varphi_{\mathrm{q}}^{+}(\mathrm{s})$ and $\varphi_{\mathrm{q}}^{-}(\mathrm{s})$.

Scale Functions

Associated with every spectrally negative Lévy process, there exists a *(q-)scale function*

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q)$$

where $\Phi(q)$ is the (largest) *positive root* of

$$\Phi(q) := \sup\{s \geq 0 : \psi(s) = q\}.$$

Moreover, $W^{(q)}(x) = 0$ on $(-\infty, 0)$.

Fluctuations

We define the *first down-* and *up-crossing times*, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X_t \leq a\},$$
$$\tau_b^+ := \inf\{t \geq 0 : X_t \geq b\}$$

for any $0 \leq a < x < b$. Then we have, for every $0 \leq x < b$,

$$\mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)},$$
$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}}\right] = Z^{(q)}(x) - Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)}$$

where $Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy$ for every $x \geq 0$.

Fluctuation for reflected processes

- $L_t^a := \sup_{0\le s\le t}(X_s - a) \vee 0,\ t \ge 0$,
- $\nu_a := \inf\{t > 0 : U_t^a < 0\}$ is the time of ruin of the reflected process $U_t^a := X_t - L_t^a$.
- We have

$$\mathbb{E}^x\Big[\int_0^{\nu_a} e^{-qt} dL_t^a\Big] = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}, \quad 0 \le x \le a.$$

Resolvent & Compensation

- For any measurable function h, we have

$$\mathbb{E}^x\left[\int_0^{\tau_0^-} e^{-qt}h(X_t)dt\right] = W^{(q)}(x)\int_0^\infty e^{-\Phi(q)y}h(y)dy - \int_0^x W^{(q)}(x-y)h(y)dy.$$

- Given a random time-space function $\phi = \phi(t,x)[\omega]$,

$$\mathbb{E}\left[\int_{[0,t]}\int_{\mathbb{R}}\phi(s,x)N(ds\times dx)\right] = \mathbb{E}\left[\int_{[0,t]}\int_{\mathbb{R}}\phi(s,x)ds\nu(dx)\right]$$

  if
  - for each $t \geq 0$, $\phi(t,x)[\omega]$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable,
  - for each $x \in \mathbb{R}$, $\phi(t,x)[\omega]$ is a.s. left continuous.

Overshoots and Undershoots

For all $\mathrm{B} \in \mathcal{B}(0,\infty)$ and $\mathrm{A} \in \mathcal{B}(-\infty,0)$,

$$\mathbb{E}^{x}\left[e^{-q\tau_0^-}\mathbf{1}_{\{X_{\tau_0^-}\in B,\, X_{\tau_0^- -}\in A,\, \tau_0^-<\infty\}}\right]$$
$$= \int_0^\infty \overline{\Pi}(du)\left\{W^{(q)}(x)\int_{B\cap(A+u)} e^{-\Phi(q)y}dy - \int_{B\cap(A+u)} dy W^{(q)}(x-y)\right\}.$$

Part 1: Optimal Stopping and Singular Control

Optimal Stopping

- Want to maximize or minimize

$$v^{\tau}(x) := \mathbb{E}_x\Big[\int_0^{\tau} e^{-qt} f(X_t)dt + e^{-q\tau} g(X_{\tau})\Big],$$

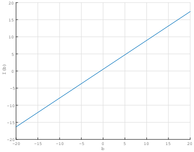
  over all stopping times τ.
- Examples include
  - American options
  - Change point detection
  - Sequential hypothesis testing
  - Secretary problems ...

When threshold strategy is expected to be optimal

- Suppose we can compute for τ_b^- (the first time X up/down-crosses b)

$$v_b(x) := \mathbb{E}_x \Big[\int_0^{\tau_b^-} e^{-qt} f(X_t) dt + e^{-q\tau_b^-} g(X_{\tau_b^-}) \Big].$$

- Guess via Smooth/continuous fit
  - choose b such that v_b is smooth/continuous at b
  - typically the condition is equivalent to the first order condition
- If lucky, we obtain a monotone function:



- Its root becomes the optimal threshold level.

Regularity and Smooth/Continuous fit

- A point $b \in \mathbb{R}$ is regular for an open or closed set B if

$$\mathbb{P}_b\{\tau^B = 0\} = 1$$

where

$$\tau^B := \inf\{t > 0 : X_t \in B\}.$$

- Typically, we pursue a stopping region B^* (in the form (b^*, ∞) or $(-\infty, b^*)$) so that the value function v_b is
  - smooth when b^* is regular for B^*;
  - continuous when b^* is irregular for B^*.

Verification of $v^* = v_{b^*}$

To show that our candidate τ_{b^*} ($v^* = v_{b^*}$) is indeed optimal, we need

- smoothness - smooth enough to apply (a version of) Ito's formula
  - already done by how b^* was chosen by smooth/continuous fit.
- variational inequality
  (i) $v^*(x) \geq g(x)$ for all $x \in \mathbb{R}$,
  (ii) $(\mathcal{L} - q)v^*(x) + f(x) = 0$ for all $x \notin B^*$,
  (iii) $(\mathcal{L} - q)v^*(x) + f(x) < 0$ for all $x \in B^*$.

  where $\mathcal{L}$ is the infinitesimal generator.
- some localizing arguments
  - v^* should not grow too rapidly in the tail – we want to take limits inside expectation)

How Verification Works

1. Ito's lemma + smoothness/continuity gives

$$e^{-qt}v^*(X_t) - v^*(X_0) = \int_0^t e^{-qs}(\mathcal{L} - q)v^*(X_{s-})ds + M_t,$$

   where M is a local martingale.

2. With $T_m := \inf\{t > 0 \ : \ |X_t| > m\}$,

$$\begin{aligned} v^*(x) &\geq \mathbb{E}^x\left[e^{-q(t\wedge\tau\wedge T_m)}v^*(X_{t\wedge\tau\wedge T_m}) + \int_0^{t\wedge\tau\wedge T_m} e^{-qs}f(X_s)ds\right] \\ &\geq \mathbb{E}^x\left[e^{-q(t\wedge\tau\wedge T_m)}g(X_{t\wedge\tau\wedge T_m}) + \int_0^{t\wedge\tau\wedge T_m} e^{-qs}f(X_s)ds\right] \end{aligned}$$

3. Finally take $t, m \uparrow \infty$.

Singular Control

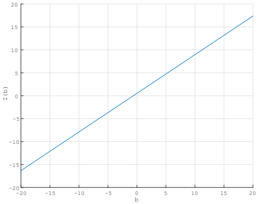
- We want to maximize/minimize

$$v^{\pi}(x) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-qt} f(X_t \pm U_t^{\pi})dt \pm \int_{[0,\infty)} e^{-qt} dU_t^{\pi}\Big].$$

- Examples include
  - Optimal dividends
  - Inventory models

Singular Control: Guessing

- We pursue a reflection barrier b^* so that the process stays on $[b^*, \infty)$ or $(-\infty, b^*]$.
- Control region B^* is its complement.
- We obtain b^* so that the expected NPV value functional v_{b^*} is
  - twice-differentiable when b^* is regular for B^*;
  - differentiable when b^* is irregular for B^*.
- If lucky, we obtain a monotone function:



- Its root becomes the optimal threshold level.

Singular Control: Verifying

To show that our candidate $v^* = v_{b^*}$ is indeed optimal, we need

- smoothness - smooth enough to apply (a version of) Ito's formula
  - already done by how b^* was chosen by the smoothness condition.
- variational inequality

  (i) ${v^*}'(x) \geq 1$ for all $x \in \mathbb{R} \backslash B^*$,

  (ii) ${v^*}'(x) = 1$ for all $x \in B^*$,

  (iii) $(\mathcal{L} - q)v^*(x) + f(x) = 0$ for all $x \in \mathbb{R} \backslash B^*$,

  (iv) $(\mathcal{L} - q)v^*(x) + f(x) \leq 0$ for all $x \in B^*$.

  where $\mathcal{L}$ is the infinitesimal generator.
- some localizing arguments
  - v^* should not grow too rapidly in the tail – we want to take limits inside expectation)

Example: part of Y. (2014)

Suppose we want to maximize

$$u^{\tau}(x) := \mathbb{E}_x\left[\int_0^{\tau} e^{-rt} f(X_t)dt + e^{-r\tau} g(X_{\tau}) 1_{\{\tau<\infty\}}\right],$$

where

- $g(x) = K - bx - \sum_{i=1}^{N} c_i e^{a_i x}$ for some constants $K \in \mathbb{R}$, $b \geq 0$ and $c_i, a_i > 0$, $1 \leq i \leq N$, $N \geq 0$,
- $f(\cdot)$ is continuous, piecewise differentiable, and increasing (and some condition on its tail).

Spectrally Negative Lévy Processes

- Let X be a spectrally negative Lévy process with a Laplace exponent:

$$\begin{aligned}\psi(s) &:= \log \mathbb{E}\left[e^{sX_1}\right] \\ &= cs + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} (e^{sz} - 1 - sz1_{\{-1<z<0\}})\nu(dz),\end{aligned}$$

  such that $\int_{(-\infty,0)} (1 \wedge z^2)\nu(dz) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} z\,\nu(dz) < \infty$.
- We exclude the case X is a subordinator.

Scale Functions

- Laplace exponent $\psi(s) = \log \mathbb{E}\left[e^{sX_1}\right]$.
- Fix any $q > 0$, there exists a function called the q-scale function

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W^{(q)}(x)dx = \frac{1}{\psi(s) - q}, \qquad s > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

- Define

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)dy, \quad \overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(y)dy.$$

Spectrally Negative Case

$$
\begin{aligned}
u_A(x) &:= \mathbb{E}_x\left[\int_0^{\tau_A} e^{-qt} f(X_t)\,dt + e^{-q\tau_A} g(X_{\tau_A}) 1_{\{\tau_A<\infty\}}\right] \\
&= K\Big(Z^{(q)}(x-A) - \frac{q}{\Phi(q)} W^{(q)}(x-A)\Big) \\
&- \sum_{i=1}^{N} c_i e^{a_i x}\Big(Z_{a_i}^{(q-\psi(a_i))}(x-A) - \frac{q-\psi(a_i)}{\Phi(q)-a_i} W_{a_i}^{(q-\psi(a_i))}(x-A)\Big) \\
&- b\Big[\overline{Z}^{(q)}(x-A) + \Big(A - \frac{\psi'(0+)}{q}\Big) Z^{(q)}(x-A) + \frac{\psi'(0+)}{q} \\
&- \frac{q - \psi'(0+)\Phi(q) + qA\Phi(q)}{\Phi(q)^2} W^{(q)}(x-A)\Big] \\
&+ W^{(q)}(x-A)\int_0^{\infty} e^{-\Phi(q)y} f(y+A)\,dy \\
&- \int_A^x W^{(q)}(x-y) f(y)\,dy.
\end{aligned}
$$

Continuous/smooth fit

- For SN Lévy process of bounded variation, A is irregular for $(-\infty, A)$ – we apply continuous fit.
- For SN Lévy process of unbounded variation, A is regular for $(-\infty, A)$ – we apply smooth fit.
- Both turn out to be equivalent to $\Lambda(A) = 0$ where

$$\Lambda(A) := -\frac{q}{\Phi(q)}K + b\Big(\frac{q}{\Phi(q)^2} + \frac{qA - \psi'(0+)}{\Phi(q)}\Big) + \sum_{i=1}^{N} c_i e^{a_i A} \frac{q - \psi(a_i)}{\Phi(q) - a_i} + \int_A^x W^{(q)}(x-y) f(y) dy.$$

- $\Lambda(A)$ is continuous and increasing.
- If $\lim_{A \downarrow -\infty} \Lambda(A) < 0 < \lim_{A \uparrow \infty} \Lambda(A)$, there exists a unique root $A^* \in \mathbb{R}$ such that $\Lambda(A^*) = 0$.
- Otherwise, let $A^* = -\infty$ if $\lim_{A \downarrow -\infty} \Lambda(A) \geq 0$ and let $A^* = \infty$ if $\lim_{A \uparrow \infty} \Lambda(A) \leq 0$.

Value functions

- Candidate Value function becomes

$$\begin{aligned} u_{A^*}(x) =& \, KZ^{(q)}(x - A^*) \\ & -b\Big[\overline{Z}^{(q)}(x - A^*) + \big(A^* - \frac{\psi'(0+)}{q}\big)Z^{(q)}(x - A^*) + \frac{\psi'(0+)}{q}\Big] \\ & - \sum_{i=1}^{N} c_i e^{a_i x} Z_{a_i}^{(q-\psi(a_i))}(x - A^*) - \int_{A^*}^{x} W^{(q)}(x - y) f(y) dy. \end{aligned}$$

- Verification
  (i) $v^*(x) \geq g(x)$ for all $x \in \mathbb{R} \longrightarrow$ easy
  (ii) $(\mathcal{L} - q)v^*(x) + f(x) = 0$ for all $x \in (A^*, \infty) \longrightarrow$ easy
  (iii) $(\mathcal{L} - q)v^*(x) + f(x) < 0$ for all $x \in (-\infty, A^*) \longrightarrow$ hard.
- For (i), observe smooth/continuous fit $\Leftrightarrow$ first order condition w.r.t. A.
- For (ii), use $(\mathcal{L} - q)Z^{(q)}(x) = (\mathcal{L} - q)(\overline{Z}^{(q)}(x) + \psi'(0+)/q) = 0$.

Optimal Dividend

- A strategy $\pi = \{L_t^\pi, t \geq 0\}$ is a nondecreasing and adapted process starting at zero.
- A controlled risk process is the difference:

$$U_t^\pi := X_t - L_t^\pi, \quad t \geq 0.$$

- Time of ruin: $\sigma^\pi := \inf\{t > 0 : U_t^\pi < 0\}$.
- We want to maximize, for $q > 0$,

$$v_\pi(x) := \mathbb{E}_x\left[\int_0^{\sigma^\pi} e^{-qt} dL_t^\pi\right],$$

over the set of all strategies Π satisfying $\Delta L_t^\pi \leq U_{t-}^\pi + \Delta X_t$ for all $t \leq \sigma^\pi$ a.s.

- We want to obtain the value function:

$$v(x) := \sup_{\pi \in \Pi} v_\pi(x), \quad x \geq 0.$$

Solution Procedures

- We follow a classical approach "guess" and "verify".
- Guess that an optimal strategy is a barrier strategy (reflected Lévy process) $\pi_a := \{L_t^a; t \leq \sigma_a\}$ in the form:

$$L_t^a := \sup_{0 \leq s \leq t} (X_s - a) \vee 0,$$
$$U_t^a := X_t - L_t^a,$$

with the corresponding ruin time $\sigma_a := \inf \{t > 0 : U_t^a < 0\}$.

- Choose the value of a using some smoothness condition.
- Verify that

$$v_a(x) := \mathbb{E}_x \left[\int_0^{\sigma_a} e^{-qt} dL_t^a \right] \geq \sup_{\pi \in \Pi} v_\pi(x).$$

SN case: Avram et al. (2004) etc

- As in Avram, Palmowski & Pistorius (2004)

$$v_a(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}, & 0 \le x \le a, \\ (x-a) + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)}, & x > a, \end{cases}$$

- a is regular for (a, ∞) – twice-differentiability should hold for a – or $W^{(q)\prime\prime}(a) = 0$.
- It turned out that optimality is not guaranteed, but Loeffen (2009) obtained a sufficient condition on the Lévy measure.

SP case: Bayraktar et al. (2013)

- Unlike the SN case, the SP case is straightforward.

$$L_t^a := \sup_{0 \le s \le t} (X_s - a) \vee 0 \quad \text{and} \quad U_t^a := X_t - L_t^a.$$

- By an Application of Avram, Palmowski, Pistorius (2004),

$$v_a(x) = \begin{cases} -k(a-x) + \frac{Z^{(q)}(a-x)}{Z^{(q)}(a)} k(a), & 0 \le x \le a, \\ \frac{k(a)}{Z^{(q)}(a)}, & x \ge a, \end{cases}$$

where

$$k(y) := \int_0^y Z^{(q)}(z) dz - \frac{1}{\Phi(q)} Z^{(q)}(y) + \frac{\psi'(0+)}{q}.$$

- X is of bounded var. $\rightarrow$ a is irregular for (a, ∞) $\rightarrow$ pursue C^1.
- X is of unbounded var. $\rightarrow$ a is regular for (a, ∞) $\rightarrow$ pursue C^2.

SP case: Bayraktar et al. (2013)

We will denote our candidate barrier level by

$$a^* = \begin{cases} \left(\overline{Z}^{(q)}\right)^{-1}\left(\frac{|\psi'(0+)|}{q}\right) > 0 & \text{if } \psi'(0+) < 0, \\ 0 & \text{if } \psi'(0+) \geq 0, \end{cases}$$

which is well-defined because $\overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z)dz$ is monotone.

Theorem (Bayraktar, Kyprianou & Y. (Astin Bull., 2013))

We have

$$v_{a^*}(x) := \sup_{\pi \in \Pi} v_\pi(x), \quad x \geq 0,$$

where

$$v_{a^*}(x) = \begin{cases} -\overline{Z}^{(q)}(a^* - x) - \frac{\psi'(0+)}{q}, & \text{if } \psi'(0+) > 0, \\ x, & \text{if } \psi'(0+) \leq 0. \end{cases}$$

Part 2:
Two-sided Singular Control

Model

- $(\Omega, \mathcal{F}, \mathbb{P})$ hosting a *spectrally negative Lévy process* X.
- An admissible strategy $\pi := \{(U_t^\pi, D_t^\pi); t \geq 0\}$ – nondecreasing, right-continuous and adapted processes such that $U_{0-}^\pi = D_{0-}^\pi = 0$ and $\mathbb{E}_x\Big[\int_{[0,\infty)} e^{-qt}(dU_t^\pi + dD_t^\pi)\Big] < \infty$.
- We want to minimize

$$v^\pi(x) := \mathbb{E}_x\Big[\int_0^\infty e^{-qt} f(Y_t^\pi) dt + \int_{[0,\infty)} e^{-qt}\,(C_U dU_t^\pi + C_D dD_t^\pi)\Big],$$

with the controlled process

$$Y_t^\pi := X_t + U_t^\pi - D_t^\pi,$$

for the case X is a spectrally negative Lévy process.

Assumptions

1. The unit proportional costs C_U and C_D can be negative but must satisfy

$$C_U + C_D > 0.$$

2. We assume that $\mathbb{E}[X_1] = \psi'(0+) \in (-\infty, \infty)$.
3. About f (common assumptions as in Bensoussan et al. (2005))
   - convex (can be generalized slightly);
   - grows (or decreases) at most polynomially;
   - There exists a number $\bar{a} \in \mathbb{R}$ such that the function

$$\tilde{f}(x) := f(x) + C_U q x, \quad x \in \mathbb{R},$$

   is increasing on $(\bar{a}, \infty)$ and is decreasing on $(-\infty, \bar{a})$.

Double reflection strategies

- A doubly reflected Lévy process given by

$$Y_t^{a,b} := X_t + U_t^{a,b} - D_t^{a,b}, \quad t \geq 0, \ a < b,$$

  which is reflected at two barriers a and b so as to stay on the interval $[a, b]$.
- The corresponding NPV of costs becomes

$$v_{a,b}(x) = \frac{\Gamma(a,b)}{qW^{(q)}(b-a)} Z^{(q)}(x-a) - C_U R^{(q)}(x-a) + \frac{f(a)}{q} - \int_a^x \overline{W}^{(q)}(x-y) f'(y) dy.$$

  For $x \geq b$, we have $v_{a,b}(x) = v_{a,b}(b) + C_D(x-b)$.
- Here

$$R^{(q)}(y) := \overline{Z}^{(q)}(y) + \frac{\psi'(0+)}{q}, \quad y \in \mathbb{R}.$$

Selection of a and b

- Define

$$\begin{aligned}\Gamma(a,b) &:= C_D + C_U Z^{(q)}(b-a) + f(b)W^{(q)}(0) + \\ &\qquad \int_a^b f(y) W^{(q)\prime}(b-y)dy - W^{(q)}(b-a)f(a), \\ \gamma(a,b) &:= \frac{\partial}{\partial b}\Gamma(a,b-).\end{aligned}$$

- We see that the values of (a,b) such that $\Gamma(a,b)$ and $\gamma(a,b)$ vanish simultaneously attain smoothness.

Selection of a and b (Cont'd)

- Taking a derivative and then limits

$$v'_{a,b}(b-) = C_D \quad \text{and} \quad v'_{a,b}(a+) = \frac{\Gamma(a,b)}{W^{(q)}(b-a)} W^{(q)}(0) - C_U.$$

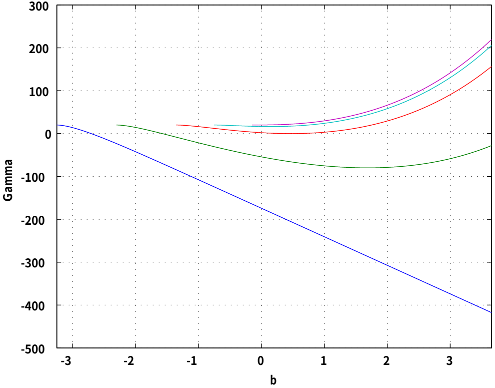
- Taking another derivative and then limits

$$v''_{a,b}(b-) = \frac{\Gamma(a,b)}{W^{(q)}(b-a)} W^{(q)\prime}((b-a)-) - \gamma(a,b),$$
$$v''_{a,b}(a+) = \frac{\Gamma(a,b)}{W^{(q)}(b-a)} W^{(q)\prime}(0+) - \tilde{f}'(a+) W^{(q)}(0).$$

- Recall also that $W^{(q)}(0) = 0$ iff X is of unbounded variation.
- If $\frac{\Gamma(a,b)}{W^{(q)}(b-a)} W^{(q)\prime}((b-a)-) = \gamma(a,b) = 0$, then $v_{a,b}$ is
  - differentiable (resp. twice-differentiable) at a when X is of bounded (resp. unbounded) variation.
  - it is twice-differentiable at b.

Existence of a^* and b^*

Plots of $b \mapsto \Gamma(a, b)$ for five fixed values of a – the red one is what we want.



Candidate Thresholds

There exist a^* and b^* such that $\Gamma(a^*, x) \geq 0$ for all $x \in [a^*, \infty)$ and either **Case 1** or **Case 2** defined below holds.

Case 1 $d^* < u^*$ and

$$\Gamma(a^*, b^*) = 0.$$

Moreover, if $\gamma(a^*, b^*)$ is continuous at u^* then we also have that $\gamma(a^*, b^*) = 0$.

Case 2 $a^* \in \mathbb{R}$ and $b^* = \infty$ and

$$\lim_{b \to \infty} \frac{\Gamma(a^*, b)}{W^{(q)}(b - a^*)} = 0.$$

Verification of optimality

With our choice of (a^*, b^*),

$$v_{a^*,b^*}(x) = -c_R\Big(\frac{\psi'(0+)}{q} + x\Big) + \frac{\tilde{f}(a^*)}{q} Z^{(q)}(x - a^*) - \int_a^x W^{(q)}(x-y)\tilde{f}'(y)dy.$$

Verification of optimality requires

1. $-C_U \leq v'_{a^*,b^*}(x) \leq C_D$ for all $x \in \mathbb{R}$.
2. $(\mathcal{L} - q)v_{a^*,b^*}(x) + f(x) \geq 0$ for all $x > b^*$.
3. $(\mathcal{L} - q)v_{a^*,b^*}(x) + f(x) = 0$ for $a^* < x < b^*$.
4. $(\mathcal{L} - q)v_{a^*,b^*}(x) + f(x) \geq 0$ for $x \leq a^*$.

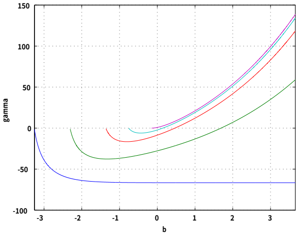
Verification of optimality (Cont'd)

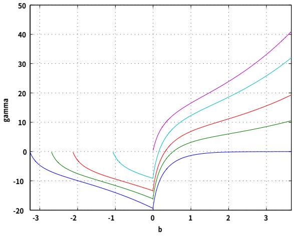
- Hard ones to prove are:
  1. $-C_U \leq v'_{a^*,b^*}(x)$ for all $x \in (a^*, b^*)$ – simple under the convexity of f,
  2. $(\mathcal{L} - q)v_{a^*,b^*}(x) + f(x) \geq 0$ for all $x > b^*$ – six page proof;

  others hold even w/o the convexity of f.
- We use the convexity of f and

$$v'_{a^*,b^*}(x) = -\Gamma(a^*, x) + C_D, \quad a^* \leq x \leq b^*.$$

- Plots of $\Gamma(a, x)$ and $\gamma(a, x) = \partial\Gamma(a, x)/\partial x$:





Numerical results

- The β-family introduced by Kuznetsov (AAP, 2010):

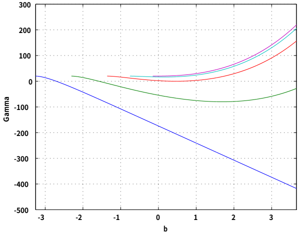
$$\psi(z) = \hat{\delta} z + \frac{1}{2}\sigma^2 z^2 + \frac{\varpi}{\beta}\left\{ B(\alpha + \frac{z}{\beta}, 1-\lambda) - B(\alpha, 1-\lambda) \right\}$$

  for some $\hat{\delta} \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $\varpi \geq 0$, $\lambda \in (0,3)\backslash\{1,2\}$ and the beta function $B(x,y)$.
- ψ is rational and hence can be inverted to obtain an analytical form of the scale function.

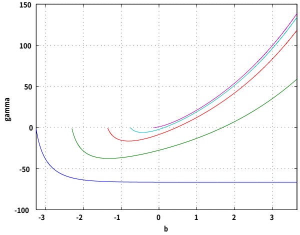
Quadratic Case

Suppose the running cost function is $f \equiv f_Q$ where

$$f_Q(x) := x^2, \quad x \in \mathbb{R}.$$

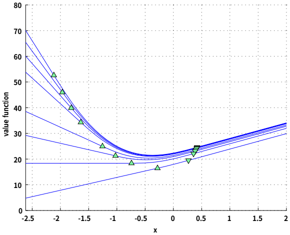


$\Gamma(a, \cdot)$

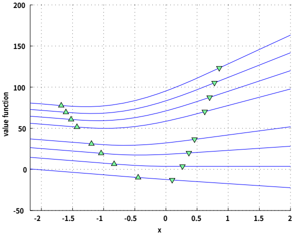


$\gamma(a, \cdot)$

Quadratic Case (Cont'd)



with respect to $\mathrm{C_U}$

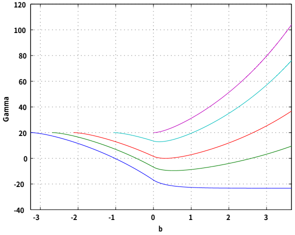


with respect to $\mathrm{C_D}$

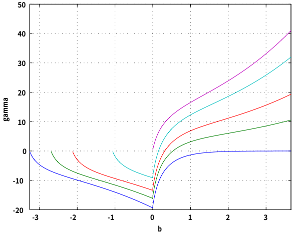
Linear Case

Suppose the running cost function is $f \equiv f_L$ where

$$f_L(x) := |x|, \quad x \in \mathbb{R}.$$

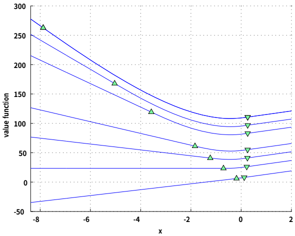


$\Gamma(a, \cdot)$

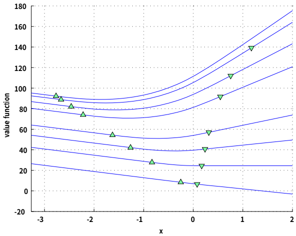


$\gamma(a, \cdot)$

Linear Case (Cont'd)



with respect to $\mathrm{C_U}$



with respect to $\mathrm{C_D}$

Part 3:
Impulse Control

Inventory Models with Fixed Costs

1. Uncontrolled surplus: $X_t, t \geq 0$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
2. An *(ordering) policy*

$$\pi := \Big\{ L_t^\pi = \sum_{i: T_i^\pi \leq t} u_i^\pi; t \geq 0 \Big\}$$

in the form of an impulse control $(T_1^\pi, u_1^\pi; T_2^\pi, u_2^\pi; \cdots)$ where

- $\{T_i; i \geq 1\}$ is an increasing sequence of $\mathbb{F}$-stopping times and
- $u_i > 0$ is an $\mathcal{F}_{T_i}$-measurable random variable for $i \geq 1$.

3. Corresponding to every policy π, the (controlled) surplus process is

$$U_t^\pi := X_t + L_t^\pi, \quad t \geq 0.$$

4. The problem is to minimize the total expected cost:

$$v_\pi(x) := \mathbb{E}_x \Big[\int_0^\infty e^{-qt} f(U_t^\pi) dt + \sum_{i=1}^\infty e^{-qT_i^\pi} (K + C u_i^\pi) \Big].$$

The Model of X

- Compound Poisson models
- Brownian Motion models
- Sum of Compound Poisson and Brownian Motion
  - Bensoussan, Liu & Sethi (SICON, 2005), Benkherouf & Bensoussan (SICON, 2009).
- Spectrally negative Lévy models (this talk) – a general Lévy process with only positive jumps that is not a negative of a subordinator.

Spectrally Negative Lévy Processes

- Let X be a spectrally negative Lévy process with a Laplace exponent:

$$\psi(s) := \log \mathbb{E}\left[e^{sX_1}\right] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} (e^{sz} - 1 - sz1_{\{-1<z<0\}})\nu(dz),$$

such that $\int_{(-\infty,0)}(1 \wedge z^2)\nu(dz) < \infty$.

- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} z\,\nu(dz) < \infty$.
- We exclude the case X is a subordinator.
- We also assume $\mu := \mathbb{E}[X_1] \in (-\infty, \infty)$.

The Problem Revisited

1. Uncontrolled surplus: $X_t, t \geq 0$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
2. An *(ordering) policy*

$$\pi := \left\{ L_t^\pi = \sum_{i: T_i^\pi \leq t} u_i^\pi; t \geq 0 \right\}$$

in the form of an impulse control $(T_1^\pi, u_1^\pi; T_2^\pi, u_2^\pi; \cdots)$ where

- $\{T_i; i \geq 1\}$ is an increasing sequence of $\mathbb{F}$-stopping times and
- $u_i > 0$ is an $\mathcal{F}_{T_i}$-measurable random variable for $i \geq 1$.

3. Corresponding to every policy π, the (controlled) surplus process is

$$U_t^\pi := X_t + L_t^\pi, \quad t \geq 0.$$

4. The problem is to minimize the total expected cost:

$$v_\pi(x) := \mathbb{E}_x \Big[\int_0^\infty e^{-qt} f(U_t^\pi) dt + \sum_{i=1}^\infty e^{-qT_i^\pi} (K + C u_i^\pi) \Big].$$

Assumptions

The assumptions are the same as those in Bensoussan, Liu & Sethi (SICON, 2005), Benkherouf & Bensoussan (SICON, 2009).

Assumption

$g(y) := Cy + K,\ y > 0$, *for some unit cost of the item* $C \in \mathbb{R}$ *and fixed ordering cost* $K > 0$. *We shall study the case* $K = 0$ *separately.*

Assumption

1. f *is a piecewise continuously differentiable function with* $f(0) = 0$.
2. *There exists a number* a *such that the function*

$$\tilde{f}(x) := f(x) + Cqx, \quad x \in \mathbb{R},$$

is increasing on (a, ∞) *and decreasing and convex on* $(-\infty, a)$.

3. *There exist a* $c_0 > 0$ *and an* $x_0 \geq a$ *such that* $\tilde{f}'(x) \geq c_0$ *for* $x \geq x_0$.

The (s, S)-Policy

- For $-\infty < s < S < \infty$, an (s, S)-policy, $\pi_{s,S} := \{L_t^{s,S}; t \geq 0\}$, brings the level of the surplus process $U^{s,S} := X + L^{s,S}$ up to S whenever it goes below s, with the corresponding NPV:

$$v_{s,S}(x) := \mathbb{E}_x\Big[\int_0^\infty e^{-qt} f(U_t^{s,S}) dt + \sum_{0 \leq t < \infty} e^{-qt} g(\Delta L_t^{s,S}) 1_{\{\Delta L_t^{s,S} > 0\}}\Big].$$

- We aim to prove that the (s^*, S^*)-policy is optimal for some $-\infty < s^* < S^* < \infty$. Toward this end,
  - Write $v_{s,S}$ analytically using the scale function.
  - Choose the value of s^* and S^* using some smoothness condition.
  - Verify that the V_{s^*,S^*} satisfies QVI (quasi-variational inequality)

Computation of $v_{s,S}(x)$

1. We shall first rewrite

$$v_{s,S}(x) := \mathbb{E}_x\Big[\int_0^\infty e^{-qt} f(U_t^{s,S})dt + \sum_{0\le t<\infty} e^{-qt} g(\Delta L_t^{s,S}) 1_{\{\Delta L_t^{s,S}>0\}}\Big],$$

in terms of the scale function.

2. By the strong Markov property, it must satisfy,

$$v_{s,S}(x) = \mathbb{E}_x\Big[\int_0^{\tau_s^-} e^{-qt} f(X_t)dt\Big] + \mathbb{E}_x\left[e^{-q\tau_s^-}(C(S - X_{\tau_s^-}) + K)\right] + \mathbb{E}_x\left[e^{-q\tau_s^-}\right] v_{s,S}(S).$$

3. Solving for $x = S$ gives $v_{s,S}(S)$ once we know these expectations.

Computation of $v_{s,S}(x)$

- Instead, we write

$$\begin{aligned}\tilde{v}_{s,S}(x) &:= v_{s,S}(x) + Cx \\ &= k(s,x) + \left(1 - \frac{q}{\Phi(q)}\overline{\Theta}^{(q)}(x-s)\right)\tilde{v}_{s,S}(S), \quad x > s,\end{aligned}$$

with $\overline{\Theta}^{(q)}(x) := W^{(q)}(x) - \Phi(q)\overline{W}^{(q)}(x)$ and

$$\begin{aligned}k(s,x) := \mathbb{E}_x\Big[\int_0^{\tau_s^-} e^{-qt} f(X_t)dt\Big] - C\mathbb{E}_x\left[e^{-q\tau_s^-}X_{\tau_s^-}\right] \\ + K\mathbb{E}_x\left[e^{-q\tau_s^-}\right] + Cx, \quad x > s.\end{aligned}$$

- Then,

$$\tilde{v}_{s,S} := \tilde{v}_{s,S}(S) = \frac{\Phi(q)}{q}\frac{k(s,S)}{\overline{\Theta}^{(q)}(S-s)}, \quad S > s.$$

Computation of $v_{s,S}(x)$

- Define, for any measurable function h and $s \in \mathbb{R}$,

$$\Psi(s;h) := \int_0^\infty e^{-\Phi(q)y} h(y+s)dy = \int_s^\infty e^{-\Phi(q)(y-s)} h(y)dy,$$
$$\varphi_s(x;h) := \int_s^x W^{(q)}(x-y)h(y)dy, \quad x \in \mathbb{R},$$
$$\mathcal{G}(s,x) := \Phi(q)\Psi(s;\tilde{f})\overline{W}^{(q)}(x-s) + K - \varphi_s(x;\tilde{f}), \quad x > s.$$

- For any $x > s$,

$$k(s,x) = \overline{\Theta}^{(q)}(x-s)\left[\Psi(s;\tilde{f}) - \frac{q}{\Phi(q)}\left(K + \frac{C\mu}{q}\right)\right] + \mathcal{G}(s,x).$$

Choosing Candidates for (s, S)

To narrow down the candidates for (s, S), we shall choose these values so that

- the function $v_{s,S}$ is continuous/smooth enough and
- its slope at S is the same as the proportional cost C.

Lemma

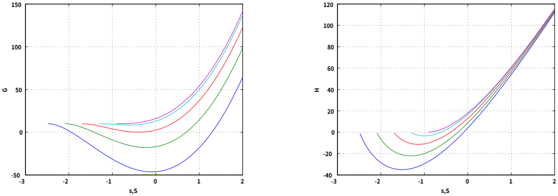
Suppose (s, S) *are such that* $\mathcal{G}(s, S) = \mathcal{H}(s, S) = 0$ *where*

$$\mathcal{H}(s, x) := \frac{\partial}{\partial x}\mathcal{G}(s, x).$$

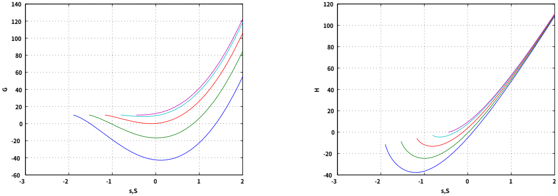
Then

1. $v_{s,S}$ *is continuous (resp. differentiable) at* s *when* X *is of bounded (resp. unbounded) variation,*
2. $\tilde{v}'_{s,S}(S) = 0$ *or equivalently* $v'_{s,S}(S) = C$*.*

Plots of $\mathcal{G}(\mathrm{s},\mathrm{S})$ and $\mathcal{H}(\mathrm{s},\mathrm{S})$



unbounded variation case



bounded variation case

Existence of (s^*, S^*)

1. Recall

$$\Psi(s; h) := \int_0^\infty e^{-\Phi(q)y} h(y+s)dy = \int_s^\infty e^{-\Phi(q)(y-s)} h(y)dy.$$

2. There exists a unique number a_0 such that $\Psi(a_0; \tilde{f}') = 0$, $\Psi(x; \tilde{f}') < 0$ if $x < a_0$ and $\Psi(x; \tilde{f}') > 0$ if $x > a_0$.
3. There exists $s^* < a_0$ and $S^* > a_0$ such that

$$s^* := \sup\left\{ s < a_0 : \inf_{S \geq s} \mathcal{G}(s, S) = 0 \right\} \quad \text{and} \quad S^* \in \arg\inf_{S \geq s} \mathcal{G}(s, S),$$

   holds with $\mathcal{H}(s^*, S^*) = \mathcal{G}(s^*, S^*) = 0$.

Verification for Optimality

Proposition

1. $(\mathcal{L} - q)v_{s^*,S^*}(x) + f(x) = 0$ *for* $x > s^*$,
2. $(\mathcal{L} - q)v_{s^*,S^*}(x) + f(x) \geq 0$ *for* $x < s^*$.

Proposition

For every $x \in \mathbb{R}$, *we have* $v_{s^*,S^*}(x) \leq K + \inf_{u \geq 0} [Cu + v_{s^*,S^*}(x+u)]$.

Theorem

The (s^*, S^*)*-policy is optimal and the value function is given by*

$$v_{s^*,S^*}(x) = \frac{\Phi(q)}{q}\Psi(s^*;\tilde{f})Z^{(q)}(x - s^*) - \varphi_{s^*}(x;\tilde{f}) - \frac{C\mu}{q} - Cx, \quad x > s^*.$$

The Case with No Fixed Costs

1. We widen the set of policies to accommodate also the processes containing diffuse components; we consider the set of $\pi := \{L^{\pi}_t; t \geq 0\}$ given by a nondecreasing, right-continuous and $\mathbb{F}$-adapted process starting at zero.
2. With $U^{\pi}_t := X_t + L^{\pi}_t$, $t \geq 0$, the problem is to compute the total costs:

$$v_{\pi}(x) := \mathbb{E}_x\Big[\int_0^{\infty} e^{-qt}\left(f(U^{\pi}_t)dt + CdL^{\pi}_t\right)\Big], \quad x \in \mathbb{R},$$

for some $C \geq 0$ and to obtain an admissible policy that minimizes it, if such a policy exists.

The Case with No Fixed Costs (Cont'd)

1. For the case $K > 0$, we have seen an (s^*, S^*)-policy is optimal for some $s^* < a_0 < S^*$.
2. Because the distance between s^* and S^* is expected to shrink as K decreases, it is a reasonable guess for the case $K = 0$ that a *barrier policy* (reflectedLévy process) with lower barrier a_0 is optimal with

$$L_t^{a_0} := \sup_{0 \leq t' \leq t} (a_0 - X_{t'}) \vee 0, \quad t \geq 0.$$

Theorem

The barrier strategy L^{a_0} is optimal and the value function is given by

$$\tilde{v}_{a_0}(x) := v_{a_0}(x) + Cx = -\frac{C\mu}{q} + \frac{Z^{(q)}(x - a_0)}{q}\tilde{f}(a_0) - \varphi_{a_0}(x; \tilde{f}).$$

Meromorphic Lévy Processes

- A class of the meromorphic Lévy process Kuznetsov, Kyprianou & Pardo (AAP, 2012) admits the Lévy measure in the form:

$$\nu(\mathrm{d}z) = \sum_{j=1}^{\infty} p_j \eta_j e^{-\eta_j z} 1_{\{z>0\}} \mathrm{d}z, \quad z \in \mathbb{R},$$

  for some $\{p_k, \eta_k; k \geq 1\}$. The equation $\psi(\cdot) = q$ has a countable negative real-valued roots $\{-\xi_{k,q}; k \geq 1\}$ that satisfy the interlacing condition:
  $\cdots < -\eta_k < -\xi_{k,q} < \cdots < -\eta_2 < -\xi_{2,q} < -\eta_1 < -\xi_{1,q} < 0$.
- The scale function can be written as

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} - \sum_{i=1}^{\infty} \frac{1}{|\psi'(-\xi_{i,q})|} e^{-\xi_{i,q}x}, \quad x \geq 0.$$

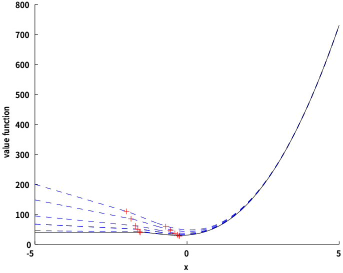
Numerical Examples

1. A spectrally negative Lévy process is said to be in the β-family if

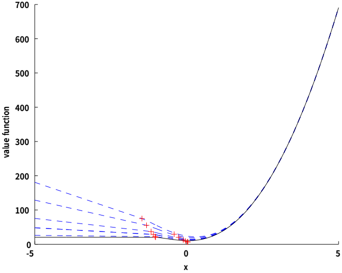
$$\psi(z) = \hat{\delta} z + \frac{1}{2}\sigma^2 z^2 + \frac{\varpi}{\beta}\left\{B(\alpha + \frac{z}{\beta}, 1-\lambda) - B(\alpha, 1-\lambda)\right\}$$

   for some $\hat{\delta} \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $\varpi \geq 0$, $\lambda \in (0,3)\backslash\{1,2\}$ and the beta function $B(x,y) := \Gamma(x)\Gamma(y)/\Gamma(x+y)$.
2. We suppose $\hat{\delta} = 0.1$, $\lambda = 1.5$, $\alpha = 3$, $\beta = 1$ and $\varpi = 0.1$. With this specification, the process has jumps of infinite activity (and of bounded variation), which is not covered in the framework of Bensoussan, Liu & Sethi (SICON, 2005).
3. We consider $\sigma = 0$ and $\sigma = 0.2$ so as to study both the bounded and unbounded variation cases.
4. We let $q = 0.03$ and for the surplus cost we consider the quadratic case $f(x) = x^2$, $x \in \mathbb{R}$.

Sensitivity w.r.t. the Proportional Cost C



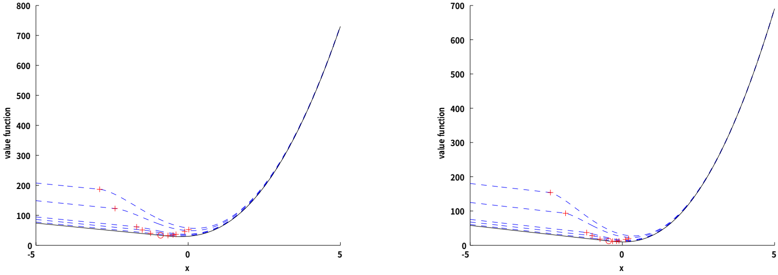
unbounded variation case ($\sigma > 0$)



bounded variation case ($\sigma = 0$)

Figure: The value functions for various values of the proportional cost C.

Sensitivity w.r.t. the Fixed Cost K



unbounded variation case ($\sigma > 0$) bounded variation case ($\sigma = 0$)

Figure: The value functions for various values of the fixed cost K.