An Algebraic Approach to the Cameron-Martin-Maruyama-Girsanov Formula

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Outline

1. Introduction
2. Overview
3. Cameron-Martin formula
4. Maruyama-Girsanov formula
5. Maruyama-Girsanov formula -algebraic structure-

Theorem 1 (Cameron-Martin)

For any $\theta \in \mathscr{H}$ and bounded measurable function $F : \mathscr{W} \to \mathbb{R}$,

$$\int_{\mathscr{W}} F(w+\theta)\gamma(\mathrm{d}w) = \int_{\mathscr{W}} F(w) e^{[\theta](w) - \frac{\|\theta\|^2_{\mathscr{H}}}{2}} \gamma(\mathrm{d}w).$$

- $(\mathscr{W}, \mathscr{B}(\mathscr{W}), \gamma)$: Wiener space,

$$\mathscr{W} = \left\{ w : [0,1] \to \mathbb{R};\, w \text{ is continuous and } w(0) = 0 \right\}$$

- $\mathscr{H}$: Cameron-Martin subspace with $\langle h_1, h_2 \rangle_{\mathscr{H}} = \langle \dot{h}_1, \dot{h}_2 \rangle_{L^2([0,1])}$,

$$\mathscr{H} = \left\{ h \in \mathscr{W};\, h \text{ is absolutely continuous and } \dot{h} \in L_2([0,1]) \right\}$$

- $[\bullet] : \mathscr{H} \to L_2(\gamma; \mathbb{R})$: Wiener integral,

$$[h](w) := \left(\int_0^1 \dot{h}(t)\mathrm{d}W(t) \right)(w) = \lim_{|\Delta| \downarrow 0} \sum_{i=0}^{n-1} \dot{h}(t_i)(w(t_{i+1}) - w(t_i)), \quad w \in \mathscr{W},$$

where W is the canonical Wiener process

Derivative on Polynomial Ring

- $\mathscr{P}$: class of polynomial functionals,

$$\mathscr{P} = \left\{ p([h_1],[h_2],\ldots,[h_n]);\, p \text{ is polynomial}, h_i \in \mathscr{H}, n \in \mathbb{N} \right\}$$

- $D_\theta : \mathscr{P} \to \mathscr{P}$ is defined by

$$D_\theta p([h]) = \langle \theta, h \rangle_{\mathscr{H}} \, p'([h])$$

for $\theta, h \in \mathscr{H}$, together with the “product rule”:

$$D_\theta(FG) = D_\theta F \cdot G + F D_\theta G, \quad F, G \in \mathscr{P}.$$

It is actually the Gâteaux derivative in the direction $\theta \in \mathscr{H}$,

$$D_\theta F(w) = \left. \frac{\mathrm{d}}{\mathrm{d}\xi} \right|_{\xi=0} F(w + \xi\theta), \quad w \in \mathscr{W}.$$

Observation

We will give a new perspective to the Cameron-Martin formula and the Maruyama-Girsanov formula by giving a “totally algebraic” proof to them.

- Observation:

$$\int_{\mathscr{W}} F(w+\theta)\gamma(\mathrm{d}w) = \int_{\mathscr{W}} e^{D_\theta} F(w)\gamma(\mathrm{d}w)$$

$$= \int_{\mathscr{W}} F(w) e^{D_\theta^*} 1 \gamma(\mathrm{d}w) = \int_{\mathscr{W}} F(w) e^{[\theta](w) - \frac{\|\theta\|_H^2}{2}} \gamma(\mathrm{d}w),$$

  where D_θ^* should be “adjoint” of D_θ.

- We need to prove (algebraically)
  - the red relation (“Taylor expansion”)
  - the purple relation (“adjointness”)
  - the blue relation (“Generating Function”)

Lemma 1 (shift operator)

For any $\theta \in \mathscr{H}$ *and* $F \in \mathscr{P}$,

$$e^{D_\theta} F(w) := \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n F(w) = F(w+\theta), \quad w \in \mathscr{W}.$$

Observe first that, for $F, G \in \mathscr{P}$,

$$e^{D_\theta} F \cdot e^{D_\theta} G = \left(\sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n F \right) \cdot \left(\sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n G \right)$$

$$= \left(F + D_\theta F + \frac{1}{2!} D_\theta^2 F + \frac{1}{3!} D_\theta^3 F + \cdots \right) \cdot \left(G + D_\theta G + \frac{1}{2!} D_\theta^2 G + \frac{1}{3!} D_\theta^3 G + \cdots \right)$$

$$= FG + \left(D_\theta F \cdot G + F D_\theta G \right) + \left(\frac{1}{2!} D_\theta^2 F \cdot G + D_\theta F \cdot D_\theta G + F \cdot \frac{1}{2!} D_\theta^2 G \right)$$

$$+ \left(\frac{1}{3!} D_\theta^3 F \cdot G + \frac{1}{2!} D_\theta^2 F \cdot D_\theta G + D_\theta F \cdot \frac{1}{2!} D_\theta^2 G + F \cdot \frac{1}{3!} D_\theta^3 G \right) + \cdots$$

$$= FG + D_\theta(FG) + \frac{1}{2!} D_\theta^2 (FG) + \frac{1}{3!} D_\theta^3 (FG) + \cdots = e^{D_\theta}(FG).$$

Here we have used Leibniz rule:

$$D_\theta^n(FG) = \sum_{k=0}^{n} \binom{n}{k} D_\theta^k F \cdot D_\theta G^{n-k},$$

The one for $n = 1$;

$$D_\theta(FG) = D_\theta F \cdot G + F D_\theta G,$$

can be obtained as easily from the definition as the one dimensional derivative, and its repeated use reach to the general formula (we can prove it by induction). The relation

$$e^{D_\theta}(FG) = e^{D_\theta} F \cdot e^{D_\theta} G$$

implies that it suffices to show for the functional $F = [h] \in \mathscr{P}$. Then,

$$e^{D_\theta} F(w) = \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n [h](w) = [h](w) + \langle \theta, h \rangle_{\mathscr{H}}$$

$$= \left(\int_0^1 \dot{h}(t)(\mathrm{d}W(t) + \dot{\theta}\mathrm{d}t) \right)(w) = \left(\int_0^1 \dot{h}(t)\mathrm{d}(W + \theta)(t) \right)(w) = F(w + \theta),$$

which competes the proof of the red relation.

Lemma 2 (adjointness)

For any $\theta \in \mathscr{H}$ and $F, G \in \mathscr{P}$,

$$\int_{\mathscr{W}} D_\theta F(w) G(w) \gamma(\mathrm{d}w) = \int_{\mathscr{W}} F(w) \overbrace{\left([\theta](w)G(w) - D_\theta G(w)\right)}^{=: D_\theta^* G(w)} \gamma(\mathrm{d}w).$$

The formula can be rewritten as

$$\langle D_\theta F, G \rangle_{L^2(\gamma;\mathbb{R})} + \langle F, D_\theta G \rangle_{L^2(\gamma;\mathbb{R})} = \langle F, [\theta] G \rangle_{L^2(\gamma;\mathbb{R})}$$

or equivalently,

$$\mathbb{E}_\gamma \left[D_\theta (FG) \right] = \mathbb{E}_\gamma \left[FG[\theta] \right].$$

By the linearity of D_θ and D_θ^*, it suffices to consider the cases when FG is in the form of

$$\prod_{k=1}^{n} H_{i_k}([\theta_{j_k}]).$$

(The details are omitted)

Remark 1 (Hermite polynomial)

- $H_n(x) := e^{\frac{x^2}{2}}(-1)^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}, n \in \mathbb{N}.$
- $H_n'(x) = nH_{n-1}(x).$
- $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$
- $\sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x) = e^{xy - \frac{y^2}{2}}.$

The third formula can be proven from the definition (the first relation) and the Leibniz rule as follows:

$$H_{n+1}(x) = e^{\frac{x^2}{2}}(-1)^n \frac{d^n}{dx^n}(xe^{-\frac{x^2}{2}}) = e^{\frac{x^2}{2}}(-1)^n \sum_{k=0}^{n} \binom{n}{k} (x)^{(k)} (e^{-\frac{x^2}{2}})^{(n-k)}$$
$$= e^{\frac{x^2}{2}}(-1)^n \left(x(e^{-\frac{x^2}{2}})^{(n)} + n(e^{-\frac{x^2}{2}})^{(n-1)} \right) = xH_n(x) - nH_{n-1}(x).$$

On the other hand, if we differentiate the first formula, we obtain

$$H_n'(x) = xH_n(x) - H_{n+1}(x),$$

which together with the third implies the second.

Lemma 3

- *For any* $\theta \in H$ *with* $||\theta||_H = 1$ *and* $n \in \mathbb{N}$,

$$(D_\theta^*)^n 1 = H_n([\theta](w)), \quad w \in \mathscr{W}.$$

- *For any* $\theta \in \mathscr{H}$,

$$e^{D_\theta^*} 1 := \sum_{n=0}^{\infty} \frac{1}{n!}(D_\theta^*)^n 1 = e^{[\theta](w) - \frac{||\theta||^2_{\mathscr{H}}}{2}}, \quad w \in \mathscr{W}.$$

We use the induction on n to prove the first formula. It is clear that

$$D_\theta^* 1 \overset{\text{Lem.2}}{=} [\theta] \overset{\text{Rem.1}}{=} H_1([\theta]).$$

Suppose that the relation holds for n. Then, noting that $\langle \theta, \theta \rangle_{\mathscr{H}} = 1$,

$$(D_\theta^*)^{(n+1)} 1 = D_\theta^* H_n([\theta]) \overset{\text{Lem.2}}{=} [\theta] H_n([\theta]) - D_\theta H_n([\theta])$$

$$\overset{\text{Rem.1}}{=} [\theta] H_n([\theta]) - n H_{n-1}([\theta]) \overset{\text{Rem.1}}{=} H_{n+1}([\theta]).$$

The second formula (the blue relation) is implied by the fourth formula in the previous Remark 1.

Theorem 1 (Cameron-Martin)

For any $\theta \in H$ *and* $F \in \mathscr{P}$,

$$\int_{\mathscr{W}} F(w+\theta)\gamma(\mathrm{d}w) = \int_{\mathscr{W}} F(w) e^{[\theta](w) - \frac{\|\theta\|_{\mathscr{H}}^2}{2}} \gamma(\mathrm{d}w).$$

$$\int_{\mathscr{W}} F(w+\theta)\gamma(\mathrm{d}w) \overset{\text{Lem.1}}{=} \int_{\mathscr{W}} e^{D_\theta} F(w)\gamma(\mathrm{d}w)$$

$$\overset{\text{Lem.2}}{=} \int_{\mathscr{W}} F(w) e^{D_\theta^*} 1 \gamma(\mathrm{d}w) \overset{\text{Lem.3}}{=} \int_{\mathscr{W}} F(w) e^{[\theta](w) - \frac{\|\theta\|_{\mathscr{H}}^2}{2}} \gamma(\mathrm{d}w).$$

□

In this talk, the domains of the operators are restricted to “polynomials” in order to concentrate on algebraic structures. However, this causes no problem since the continuity of the operators is easily checked.

Maruyama-Girsanov formula

Theorem 2 (Maruyama-Girsanov)

Let $Z \in L_2(\gamma; \mathscr{H})$. Then for any bounded measurable function $F : \mathscr{W} \to \mathbb{R}$,

$$\mathbb{E}_\gamma\left[F(W)\right] = \mathbb{E}_\gamma\left[F(W - Z)\mathscr{E}\left(\int_0^\bullet \dot{Z}(t)\mathrm{d}W(t)\right)_1\right].$$

- $\mathscr{E}(\bullet)$: Doléans-Dade exponential,

  $$\mathscr{E}(M)_t := e^{M_t - \frac{1}{2}\langle M\rangle_t}, \quad 0 \le t \le 1,$$

  where M is a locally square integrable $(\mathscr{F}_t^W)$-martingale with $M(0) = 0$

Remark 2

$\dot{Z}$ is a $(\mathscr{F}_t^W)$-predictable process with $\mathbb{E}_\gamma[\int_0^1 \dot{Z}(t)^2 \mathrm{d}t] = \mathbb{E}_\gamma[||Z||^2_{\mathscr{H}}] < \infty$. Indeed, $Z = (W(t) \circ Z)$ is a continuous $(\mathscr{F}_t^W)$-adapted process.

We define $D_Z : \mathscr{P} \to L_1(\gamma; \mathbb{R})$ corresponding to D_θ in the CM formula as

$$D_Z F(w) := \langle DF(w), Z(w) \rangle_{\mathscr{H}}, \quad w \in \mathscr{W},$$

where $D : \mathscr{P} \to \mathscr{P}(\mathscr{H})$ is the Malliavin derivative.

Lemma 4 (corresponding to the shift operator e^{D_θ})

For any $F \in \mathscr{P}$,

$$e^{D_Z} F := \sum_{n=0}^{\infty} \frac{1}{n!} D_Z^{\otimes n} F = F(W + Z).$$

Lemma 5 (corresponding to the shift density $e^{D_\theta^*} 1$)

For any $F \in \mathscr{P}$,

$$\mathbb{E}_\gamma \left[e^{L_Z} F(W) \right] \overset{\text{Itô formula}}{=} \mathbb{E}_\gamma \Bigg[F(W) \overbrace{\mathscr{E}\Big(\int_0^\bullet \dot{Z}(t) \mathrm{d}W(t) \Big)_1}^{= e^{L_Z^*} 1} \Bigg],$$

where $e^{L_Z} F = F(1 + \mathscr{E}(\int_0^\bullet \dot{Z}(t) \mathrm{d}W(t))_1) - e^{D_{-Z}} F \mathscr{E}(\int_0^\bullet \dot{Z}(t) \mathrm{d}W(t))_1$.

Theorem 2 (Maruyama-Girsanov)

Let $Z \in L_2(\gamma; \mathscr{H})$*. Then for any* $F \in \mathscr{P}$*,*

$$\mathbb{E}_\gamma[F(W)] = \mathbb{E}_\gamma\left[F(\theta - Z)\mathscr{E}\left(\int_0^\bullet \dot{Z}(t)\mathrm{d}W(t)\right)_1\right].$$

$$\mathbb{E}_\gamma\left[F(W - Z)\mathscr{E}\left(\int_0^\bullet \dot{Z}(t)\mathrm{d}W(t)\right)_1\right] \overset{\text{Lem.4}}{=} \mathbb{E}_\gamma\left[e^{D_{-Z}}F(W)\mathscr{E}\left(\int_0^\bullet \dot{Z}(t)\mathrm{d}W(t)\right)_1\right]$$

$$= \mathbb{E}_\gamma\left[F(W)\left(1 + \mathscr{E}\left(\int_0^\bullet \dot{Z}(t)\mathrm{d}W(t)\right)_1\right)\right] - \mathbb{E}_\gamma\left[e^{L_Z}F(W)\right] \overset{\text{Lem.5}}{=} \mathbb{E}_\gamma[F(W)].$$

□

This proof of Theorem 2 is "not algebraic" as it involves the Itô's formula. This means, we feel, a considerable part of the "algebraic structure" of the Maruyama-Girsanov formula is still unrevealed.

Maruyama-Girsanov formula -algebraic structure-

We consider a simple process

$$Z(w) = \sum_{i=1}^{2^m} 2^{\frac{i}{2}} Z_i(w) \mathbf{1}_{(\frac{i-1}{2^m}, \frac{i}{2^m}]}, \quad w \in \mathscr{W},$$

where $(Z_i)_{i=1}^{2^m}$ is an $(\mathscr{F}^W_{\frac{i-1}{2^m}})_{i=1}^{2^m}$-adapted process.

Lemma 6 (corresponding to the shift operator e^{D_θ} and e^{D_Z})

For any $F \in \mathscr{P}_{\text{Haar}} = \{p([h_1], [h_2], \dots, [h_n]) \in \mathscr{P}; (h_i)_{i=1}^n$: Haar system$\}$,

$$e^{D_{Z_{2^m}}} e^{D_{Z_{2^m-1}}} \cdots e^{D_{Z_1}} F = F\left(W + \int_0^{\bullet} Z(t)\mathrm{d}t\right),$$

where for any $i = 1, 2, \dots, 2^m$,

$$e^{D_{Z_i}} = \sum_{n=0}^{\infty} \frac{1}{n!} Z_i^n D_{\theta_i}^n, \quad \theta_i = 2^{\frac{n}{2}} \int_0^{\bullet} \mathbf{1}_{(\frac{i-1}{2^m}, \frac{i}{2^m}]}(t)\mathrm{d}t \in \mathscr{H}.$$

Lemma 7 (corresponding to the shift density $e^{D^*_\theta}1$ and $e^{L^*_Z}1$)

- *For any* $i = 1, 2, \ldots, 2^m$ *and* $\mathscr{F}^W_{\frac{i-1}{2^m}}$*-measurable function* F,

$$e^{D^*_{Z_i}}F := \sum_{n=0}^{\infty} \frac{1}{n!} Z_i^n (D^*_{\theta_i})^n F = F e^{D^*_{Z_i}} 1.$$

-
$$e^{D^*_{Z_{2^m}}} e^{D^*_{Z_{2^m-1}}} \cdots e^{D^*_{Z_1}} 1 = \mathscr{E}\left(\int_0^{\bullet} Z(t) \mathrm{d}W(t)\right)_1.$$

We can obtain the shift density without the Itô's formula.

$$D^*_{\theta_i} F \overset{\text{Lem.2}}{=} \left(2^{\frac{n}{2}} \left(W(\frac{i}{2^m}) - W(\frac{i-1}{2^m})\right) - D_{\theta_i}\right) F$$
$$\overset{\text{adapted}}{=} F 2^{\frac{m}{2}} \left(W(\frac{i}{2^m}) - W(\frac{i-1}{2^m})\right) \overset{\text{Lem.2}}{=} F D^*_{\theta_i} 1,$$

$$e^{D^*_{Z_i}} 1 = \sum_{n=0}^{\infty} \frac{1}{n!} Z_i^n (D^*_{\theta_i})^n 1 \overset{\text{Lem.3}}{=} \exp\left\{ Z_i 2^{\frac{m}{2}} \left(W(\frac{i}{2^m}) - W(\frac{i-1}{2^m})\right) - \frac{1}{2} Z_i^2 \right\}.$$

Theorem 2 (Maruyama-Girsanov)

Let Z be a simple process. Then for any $F \in \mathscr{P}_{\text{Haar}}$,

$$\mathbb{E}_\gamma[F(W)] = \mathbb{E}_\gamma\left[F(W - \int_0^\bullet Z(t)\mathrm{d}t)\mathscr{E}(\int_0^\bullet Z(t)\mathrm{d}W(t))_1\right].$$

$$\begin{aligned}\mathbb{E}_\gamma[F(W)] &= \mathbb{E}_\gamma\left[e^{D_{Z_1}}e^{D_{Z_2}}\cdots e^{D_{Z_{2m}}}e^{-D_{Z_{2m}}}e^{-D_{Z_{2m-1}}}\cdots e^{-D_{Z_1}}F\right]\\ &= \mathbb{E}_\gamma\left[e^{D_{-Z_{2m}}}e^{D_{-Z_{2m-1}}}\cdots e^{D_{-Z_1}}Fe^{D^*_{Z_{2m}}}e^{D^*_{Z_{2m-1}}}\cdots e^{D^*_{Z_1}}1\right]\\ &\overset{\text{Lem.6,7}}{=} \mathbb{E}_\gamma\left[F(W - \int_0^\bullet Z(t)\mathrm{d}t)\mathscr{E}(\int_0^\bullet Z(t)\mathrm{d}W(t))_1\right].\end{aligned}$$

□

Conclusions and References

formula	Cameron-Martin	Maruyama-Girsanov
direction	θ: constant	Z: random variable
directional derivative	$D_\theta F = \langle DF, \theta \rangle_{\mathcal{H}}$	$D_Z F = \langle DF, Z \rangle_{\mathcal{H}}$
shift operator	$e^{D_\theta} = \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n$	$e^{D_Z} = \sum_{n=0}^{\infty} \frac{1}{n!} D_Z^{\otimes n}$
shift density	$e^{D_\theta^*} 1 = e^{[\theta] - \frac{\|\theta\|_{\mathcal{H}}^2}{2}}$	$e^{L_Z^*} 1 = \mathcal{E}(\int_0^\bullet \dot{Z}(t) dW(t))_1$

The present study is largely motivated by P. Malliavin's way to look at stochastic calculus, which for example appears in Malliavin (1997) and Malliavin-Thalmaier (2005), and also by some operator calculus often found in the quantum fields theory (see e.g. Miwa-Jimbo-Date (2000)).

J. Akahori, T. Amaba and S. Uraguchi. An Algebraic Approach to the Cameron-Martin-Maruyama-Girsanov Formula, Math. J. Okayama Univ. **55** (2013), 167-190.

Thank you for your attention.