On Factorizable Markov operators on non-commutative probability spaces

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§1. Markov operator and Rota theorem

(1) Classical probability case

Definition

Let (X, μ) be a probability space. A linear map Q is called a μ-preserving Markov operator on (X, μ) if it is a positive unital normal operator from $L^\infty(X, \mu)$ into itself and it is μ-preserving, that is

$$\int_X Q(f) d\mu = \int_X f d\mu$$

for every $f \in L^\infty(X, \mu)$.

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Let Q be a μ-preserving Markov operator. Then there exists a unique μ-preserving Markov operator Q^* on (X, μ) such that

$$\int_X Q^*(f) g d\mu = \int_X f Q(g) d\mu, \qquad f, g \in L^\infty(X, \mu).$$

Rota theorem

Theorem (Rota, 1962)

Let P be a μ-preserving Markov operator on a probability space (X, μ). Then for all $p > 1$ and $f \in L^p(X, \mu)$, the sequence $(P^n(P^*)^n(f))$ converges almost surely.

(2) Noncommutative probability case

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In algebraic sense,

- Q is a positive, unital, linear map from $(L^\infty(X, \mu), \mathbb{E})$ into itself.
- That Q is μ-preserving means that $\mathbb{E} \circ Q = \mathbb{E}$.
- There exists a unique μ-preserving Markov operator Q^* from L^∞ into itself such that

$$\mathbb{E}(Q^*(f)g) = \mathbb{E}(fQ(g))$$

for all $f, g \in L^\infty$, where $\mathbb{E}(f) := \int_X f d\mu$ for all $f \in L^\infty$.

Let $(\mathcal{M}, \phi)$ and $(\mathcal{N}, \psi)$ be W^*-probability spaces, that is, pairs of von Neumann algebras and its (tracial) normal faithful state.

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Lemma

Let Q be a completely positive, unital, linear map from $(\mathcal{M}, \phi)$ into $(\mathcal{N}, \psi)$. The following conditions are equivalent.

- $\psi \circ Q = \phi$ *and* $\sigma_t^{\psi} \circ Q = Q \circ \sigma_t^{\phi}$*, for all* $t \in \mathbb{R}$*,*
- *There exists a unique completely positive, unital, linear map* $Q^* : (\mathcal{N}, \psi) \to (\mathcal{M}, \phi)$ *such that* $\phi(Q^*(y)x) = \psi(yQ(x))$ *for all* $x \in \mathcal{M}$*,* $y \in \mathcal{N}$*.*

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Definition

Let Q be a completely positive, unital, linear map from $(\mathcal{M}, \phi)$ into $(\mathcal{N}, \psi)$. Q is called a (ϕ, ψ)-Markov operator if Q satisfies that $\psi \circ Q = \phi$ and $\sigma_t^{\psi} \circ Q = Q \circ \sigma_t^{\phi}$, for all $t \in \mathbb{R}$.

Noncommutative Rota theorem in a special case

Let $(\mathcal{M}, \phi)$ be a W^*-probability space and σ_i unital endomorphisms of $\mathcal{M}$ for $i \in I$ where I is a finite set of indices.

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- Assume that $\phi \circ \sigma_i = \phi$ and $\sigma_i \circ \sigma_t^\phi = \sigma_t^\phi \circ \sigma_i$ for all $i \in I$ and $t \in \mathbb{R}$.
- Give a stochastic matrix $\{p_{ij}\}$ and a stationary distribution $\{p_i\}_{i \in I}$ with $p_i > 0$ for all $i \in I$.

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Set $\mathcal{N} := \mathcal{M}^I$ and its (normal faithful) state as follow

$$\psi(y) := \sum_{i\in I} p_i \phi(y_i), \qquad y = (y_i)_{i\in I} \in \mathcal{N}.$$

Define the following ϕ-Markov operator P from $\mathcal{N}$ into itself.

$$P(y)_i := \sum_{j\in I} p_{ij} \sigma_i(y_j), \qquad y = (y_j)_{j\in I} \in \mathcal{N}, i \in I.$$

Theorem (Noncommutative Markov chain construction (MCC), Anantharaman-Delaroche, 2006)

There exist

- *a unital* $*$*-monomorphism* $\beta : \mathcal{N} \to \mathcal{N}$ *satisfying* $\psi \circ \beta = \psi$ *and* $\sigma_t^{\psi} \circ \beta = \beta \circ \sigma_t^{\psi}$ *for all* $t \in \mathbb{R}$,
- *a unital* $*$*-monomorphism* $J_0 : \mathcal{N} \to \mathcal{N}$ *satisfying* $\psi \circ J_0 = \psi$ *and* $\sigma_t^{\psi} \circ J_0 = J_0 \circ \sigma_t^{\psi}$ *for all* $t \in \mathbb{R}$,

such that the following conditions hold:
(i) $\mathcal{N}_{\leq n}$ *and* $\mathcal{N}_{\geq n}$ *are* ψ*-invariant*
(ii) if $\mathbb{E}_{\leq n}$ *and* $\mathbb{E}_{\geq n}$ *are the corresponding conditional expectations for* $n \in \mathbb{N}$, *we have that*

$$\mathbb{E}_{\leq n} \circ J_m = J_n \circ P^{m-n}, \;\; \mathbb{E}_{\geq n} \circ J_0 = J_n \circ (P^*)^n, \qquad m \geq n,$$

where

- $J_n := \beta^n \circ J_0, \qquad n \in \mathbb{N}$
- $\mathcal{N}_{\leq n}$: von Neumann subalgebra of $\mathcal{N}$ generated by $\cup_{k \leq n} J_k(\mathcal{N})$
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Lemma (Anantharaman-Delaroche, 2006)

Let $(\mathcal{N}, \psi)$ be a W^*-probability space and $\{\mathcal{N}_n\}$ a decreasing sequence of ψ-invarant von Neumann subalgebras of $\mathcal{N}$. Let Q be a completely positive, unital contraction. Then we have

- *For $1 < p \leq \infty$ and $x \in L^p(\mathcal{N}, \psi)$, the sequence $\{Q \circ \mathbb{E}_n(x)\}$ converges to $Q \circ \mathbb{E}_\infty(x)$ bilaterally almost surely,*
- *For $2 \leq p \leq \infty$ and $x \in L^p(\mathcal{N}, \psi)$, the sequence $\{Q \circ \mathbb{E}_n(x)\}$ converges to $Q \circ \mathbb{E}_\infty(x)$ almost surely,*

where we set $\mathcal{N}_\infty := \cap \mathcal{N}_n$ and denote by $\mathbb{E}_n$ the ψ-preserving conditional expectation from $\mathcal{N}$ onto $\mathcal{N}_n$ for all $0 \leq n \leq \infty$.

From the theorem and lemma, we can show that

Corollary (Noncommutative Rota theorem, special ver., Anantharaman-Delaroche, 2006)

For $1 < p \leq \infty$ *and* $x \in L^p(\mathcal{N}, \psi)$*, the sequence* $(P^n \circ (P^*)^n(x))$ *converges bilaterally almost surely. For* $2 \leq p \leq \infty$ *the convergence hols almost surely.*

By MCC, we have that $\mathbb{E}_{\leq 0} \circ \mathbb{E}_{\geq n} \circ J_0(x) = J_0 \circ P^n \circ (P^*)^n(x)$ for all $x \in L^p(\mathcal{N}, \psi)$. And previous lemma implies the above corollary.

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$\rightarrow$ The extension of Rota theorem to the noncommutative setting is an interesting open problem.

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Definition (Anantharaman-Delaroche, 2006)

Let $P : (\mathcal{M}, \phi) \to (\mathcal{N}, \psi)$ be a (ϕ, ψ)-Markov operator. P is called to be factorizable if there exist

- a W^*-probability space $(\mathcal{L}.\xi)$,
- (ϕ, ξ)-Markov $*$-monomorphism $j_0 : (\mathcal{M}, \phi) \to (\mathcal{L}, \xi)$,
- (ψ, ξ)-Markov $*$-monomorphism $j_1 : (\mathcal{N}, \psi) \to (\mathcal{L}, \xi)$,

such that $P = j_1^* \circ j_0$.

Indeed, if P has the Noncommutative MCC, then we take $(\mathcal{L}, \xi) := (\mathcal{M}, \phi)$ and $j_i = J_i$, $(i = 0, 1)$. So that $j_1^* \circ j_0 = J_0^{-1} \circ \mathbb{E}_{\geq 0} \circ J_1 = J_0^{-1} \circ J_0 \circ P = P$.

Finally, we have that

Theorem (Anantharaman-Delaroche, 2006)

Let P be a ϕ-Markov map on a W^*-probability space $(\mathcal{M}, \phi)$. Then the following conditions are equivalent:
(i) P has the Noncommutative MCC.
(ii) P is factorizable.

Therefore every factorizable ϕ-Markov map on a W^*-probability space $(\mathcal{M}, \phi)$, satisfies the noncommutative Rota theorem.

§2. Factorizable Markov operators: View of Quantum Information Theory

M_n: the $n \times n$-matrix algebra over $\mathbb{C}$,
τ_n: the normalized trace, that is, $\tau_n(x_{ij}) := \frac{x_{11}+\cdots+x_{nn}}{n}$ for all $(x_{ij}) \in M_n$.

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Definition

A linear map $T : M_n \rightarrow M_n$ is called a quantum channel if T is unital, completely positive and trace-preserving (i.e. $\tau_n \circ T = \tau_n$).

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T is a quantum channel $\Leftrightarrow$ T is a τ_n-Markov operator on (M_n, τ_n).

Theorem (Haagerup, Musat, 2011)

Let T be a factorizable quantum channel on M_n. TFAE.
(i) T is factorizable,
(ii) T has an exact factorization through $(M_n \otimes \mathcal{M}, \tau_n \otimes \phi)$, where $(\mathcal{M}, \phi)$ is a tracial W^*-probability space, that is, there exists a unitary $u \in M_n \otimes \mathcal{M}$ such that

$$T(x) = (id_n \otimes \phi)(u^*(x \otimes 1_{\mathcal{M}})u), \qquad x \in M_n,$$

where $1_{\mathcal{M}}$ is a unit element on $\mathcal{M}$.

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Theorem (Haargerup, Musat, 2015)

Let T be a quantum channel on M_n. TFAE.
(i) it has an exact factorization through $(M_n \otimes L^\infty([0,1], dx), \tau_n \otimes \mathbb{E})$,
(ii) $T \in$conv(Aut(M_n))

$$:= \left\{ \sum_{i=1}^{d} \alpha_i ad(u_i); d \in \mathbb{N}, \alpha_i > 0 \text{ with } \sum_{i=1}^{d} \alpha_i = 1, u_i \in \mathcal{U}(M_n) \right\}.$$

Example (Haagerup, Musat, 2011)
Let $B = (b_{ij})_{1\le i,j\le 6}$ be the following 6×6 matrix and $\beta = \frac{1}{\sqrt{5}}$.

$$\begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & -\beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}.$$

Given a quantum channel T on M_6 defined by $T(x) := (b_{ij}x_{ij})$ for all $x = (x_{ij}) \in M_6$ (T is called Schur multiplier associated to B). Then T is factorizable, but it is not in $\mathrm{conv}(\mathrm{Aut}(M_6))$.

Problem (Asymptotic Quantum Birkhoff Property (AQBP), Smolin, Verstraete and Winter)

A quantum channel T on M_n is called to satisfy the AQBP if

$$\lim_{k\to\infty} d_{cb}(\otimes_{i=1}^{k} T, conv(Aut(M_{n^k})) = 0.$$

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Theorem (Haagerup, Musat, 2015)

Let T be a Schur multiplier on M_n and S a Schur multiplier on M_k. Then we have

$$d_{cb}(T\otimes S, conv(Aut(M_{nk}))) \geq \frac{1}{2} d_{cb}(T, conv(Aut(M_n))).$$

The previous example of T fails the AQBP.

§3. Completely depolarizing channel

Let $S_k : M_k \to M_k$ be a linear map defined by

$$S_k(x) := \tau_n(x)1_k, \qquad x \in M_k.$$

The linear map S_k is a quantum channel on M_k and called the completely depolarizing channel. We have that $S_k \in$conv$(\mathrm{Aut}(M_k))$.

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Theorem (Haagerup, Musat, 2015)

Let $T : M_n \to M_n$ be a quantum channel on M_n. If T has an exact factorization through $(M_n \otimes M_k, \tau_n \otimes \tau_k)$ for some positive integer k, then $T \otimes S_k \in$conv(Aut($M_n \otimes M_k$)).

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Answer: "Yes" as a partial result. In general, it is "No".

Main theorem 1

Theorem (U, 2018)

Let $T; M_n \to M_n$ be a quantum channel on M_n. If there exists a positive integer such that $T \otimes S_k = \sum_{i=1}^{d(k)} \alpha_i ad(u_i) \in$conv$(Aut(M_n \otimes M_k))$ for some positive integer $d(k)$, unitary matrices $u_1, \dots, u_{d(k)}$ and positive rational numbers $\alpha_1, \dots, \alpha_{d(k)}$ with $\sum_{i=1}^{d(k)} \alpha_i = 1$, then T has an exact factorization through $(M_n \otimes M_N.\tau_n \otimes \tau_N)$ for some positive integer N.

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Define the following set

$$I_n := \left\{ T \in \mathcal{FM}(n) \;\middle|\; \begin{array}{l} \exists k \in \mathbb{N} \text{ s.t. } T \text{ has an exact factorization through} \\ (M_n \otimes M_k, \tau_n \otimes \tau_k) \end{array} \right\},$$

where $\mathcal{FM}(n)$ is the set of all factorizable quantum channels on M_n.

Lemma

The set I_n is closed under convex combinations with rational coefficients.

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Proof.

Suppose that $T_1, \dots, T_d \in I_n$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Q}_+$ with $\sum_{i=1}^{d} \alpha_i = 1$. We claim that $\sum_{i=1}^{d} \alpha_i T_i \in I_n$. By definition of I_n, for each i we can find a unitary $u_i \in M_n \otimes M_k$ (for some common k) such that

$$T_i(x) = (id_n \otimes \tau_k)(u_i^*(x \otimes 1_k)u_i), \qquad x \in M_n.$$

Consider $\alpha_i = \frac{L_i}{L}$ for each i and we construct the following unitary matrix:

$$U := \mathrm{diag}(\overbrace{u_1, \cdots, u_1}^{L_1}, \cdots, \overbrace{u_d, \cdots, u_d}^{L_d}) \in M_n \otimes M_k \otimes M_L.$$

Then we have that $T(x) = (id_n \otimes \tau_{kL})(U^*(x \otimes 1_{kL})U)$. Hence $T \in I_n$. □

By using previous lemma, it is easy to prove Main theorem 1.

Proof of Main theorem 1.

Suppose that $T \otimes S_k = \sum_{i=1}^{d(k)} \alpha_i ad(u_i) \in$conv(Aut($M_n \otimes M_k$)) where $d(k), \alpha_i, u_i$ are the same situation of Main theorem 1. Then

$$T(x) = (id_n \otimes \tau_k)(T \otimes S_k)(x \otimes 1_k) \\ = \sum_{i=1}^{d(k)} \alpha_i (id_n \otimes \tau_k)(u_i^*(x \otimes 1_k)u_i).$$

For each i, we have that $(id_n \otimes \tau_k)(u_i^*(\cdot \otimes 1_k)u_i) \in I_n$. By previous lemma, $T \in I_n$. This means that T has an exact factorization through $M_n \otimes M_N$ for some N □

Main theorem 2

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Proof (part 1).

We define a τ_3-Markov map T_λ given by

$$T_\lambda(x) := \lambda x + (1-\lambda)u^* x u, \qquad x \in M_3,$$

where $u := \mathrm{diag}(1, i, -1)$ and $\lambda \in [0,1] \setminus \mathbb{Q}$. It is clear that $T_\lambda \otimes S_l \in$conv$(\mathrm{Aut}(M_3 \otimes M_l))$ for all l. Assume that T_λ has an exact factorization through $(M_3 \otimes M_k, \tau_3 \otimes \tau_k)$ for some k. □

Proof (part 2).

Here, T_λ is the Schur multiplier associated to the following matrix:

$$\begin{pmatrix} 1 & \lambda + (1-\lambda)i & 2\lambda - 1 \\ \lambda - (1-\lambda)i & 1 & \lambda + (1-\lambda)i \\ 2\lambda - 1 & \lambda - (1-\lambda)i & 1 \end{pmatrix}.$$

Then we can find three unitary matrices $u_0, u_1, u_2 \in M_k(\mathbb{C})$ such that $\tau_k(u_p^* u_q) = \lambda + (1-\lambda)i^{q-p}$ for all $p, q = 0, 1, 2$ by [Haagerup and Musat, 2011]. Without loss of generality we may assume that $u_0 = 1_k$ by replacing u_p by $u_0^* u_p$ for all $p = 0, 1, 2$. Then we have the linear relation:

$$u_2 = -i1_k + (1+i)u_1.$$

From the linear relation, the matrices u_1 and u_2 commute and therefore we can choose a unitary matrix $V \in M_k(\mathbb{C})$ such that $V^* u_1 V$ and $V^* u_2 V$ are diagonal. □

Proof (part 3).

Consider that u_1 has eigenvalues $e^{i\theta_1}, \cdots, e^{i\theta_k}$. Then u_2 has eigenvalues $-i + (1+i)e^{i\theta_j}$ ($j = 1, \cdots, k$) by using the above relation and the unitary matrix V. Since u_2 is unitary, we have that

$$1 = |-i + (1+i)e^{i\theta_j}|^2 = 3 - 2\sqrt{2}\sin\left(\theta_j + \frac{\pi}{4}\right), \qquad j = 1, \cdots, k.$$

This implies that u_2 has eigenvalues 1 (l times) or -1 ($k-l$ times) for some $0 \le l \le k$. Therefore we have that $\tau_k(u_2) = \frac{2l-k}{k}$. But this implies that $\lambda \in \mathbb{Q}$. Hence T_λ does not have an exact factorization through $(M_3(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for any positive integers k. □

§4. Future works

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- Let T be a factorizable Markov map on a general noncommutative probability space $(\mathcal{M}, \phi)$. Can we characterize a tensor property $T \otimes S_k \in \overline{conv}^w(\mathrm{Aut}(\mathcal{M} \otimes M_k))$?

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- Suppose that $T \in \mathcal{FM}(n)$ If we have that

$$\lim_{k \to \infty} d_{cb}(T \otimes S_k, \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_k)) = 0, \tag{1}$$

then can we find k such that $T \otimes S_k \in$conv$(\mathrm{Aut}(M_n \otimes M_k))$?

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- Let T be a factorizable Markov map on a general noncommutative probability space $(\mathcal{M}, \phi)$. Can we characterize a tensor property $T \otimes S_k \in \overline{conv}^w(\mathrm{Aut}(\mathcal{M} \otimes M_k))$?
- Suppose that $T \in \mathcal{FM}(n)$ If we have that

$$\lim_{k\to\infty} d_{cb}(T \otimes S_k, \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_k)) = 0, \quad (1)$$

then can we find k such that $T \otimes S_k \in$conv$(\mathrm{Aut}(M_n \otimes M_k))$?

$\rightarrow$ We expect to be able to approach to the Connes embedding problem which is one of the most important open problems from operator algebra theory.

Corollary (U.)

Assume that if every factorizable quantum channel T on M_n satisfies the following implication:

If T satisfies the equality (1), then there exists a positive integer k such that $T \otimes S_k \in$conv(Aut($M_n \otimes M_k$)).

Corollary (U.)

Assume that if every factorizable quantum channel T on M_n satisfies the following implication:

If T satisfies the equality (1), then there exists a positive integer k such that $T \otimes S_k \in$conv(Aut($M_n \otimes M_k$)).

Then the following conditions are equivalent.

- *Connes embedding problem has a positive answer, that is, every II_1 factor acting on some separable Hilbert space embeds into ultraproduct $\mathcal{R}^\omega$ of the hyperfinite II_1 factor $\mathcal{R}$,*
- *For all $n \in \mathbb{N}$ and $T \in \mathcal{FM}(n)$, there exists $k \in \mathbb{N}$ such that $T \otimes S_k \in \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_k))$,*
- *For all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $T \otimes S_k \in \mathrm{conv}(\mathrm{Aut}(M_n \otimes M_k))$ for all $T \in \mathcal{FM}(n)$.*

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