L^β distance between two One-dimensional Stochastic differential equations driven by a symmetric α-stable process

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Definition of Lévy process

$Z = (Z_t)_{t \geq 0}$ is a Lévy process on the probability space on $(\Omega, \mathcal{F}, \mathbb{P})$ if the following conditions are satisfied:

1. Z starts at zero, almost surely. That is, $Z_0 = 0$ a.s.
2. Z has independent increments. That is, for any $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n < \infty$, the random variables $Z_{t_1}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}$ are independent.
3. Z has stationary increments. That is, for all $0 \leq s < t < \infty$, $Z_t - Z_s$ is equal in distribution to Z_{t-s}.
4. For any $\varepsilon > 0$ and $t \in [0, \infty)$, $\lim_{h \to 0} \mathbb{P}\left(|Z_{t+h} - Z_t| > \varepsilon\right) = 0$.
5. There exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, $Z_t(\omega)$ is a càdlàg function on $[0, T]$. That is, $t \mapsto Z_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits in $(0, \infty)$.

Symmetric α-stable processes

A characteristic function of Lévy process Z is given by Lévy triplet (b, A, ν), where $b \in \mathbb{R}$, $A \geq 0$ and ν is σ finite measure with $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge |x|^2)\nu(dx) < \infty$,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta Z_t}] = \exp\left(t\{\mathrm{i}b\theta - \frac{1}{2}A\theta^2 + \int_{\mathbb{R}\setminus\{0\}}\left(\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x 1_{\{|x|<1\}}\right)\nu(dx)\}\right).$$

A Lévy process is called a symmetric α-stable for every $\alpha \in (0, 2)$ if $b = A = 0$ and

$$\nu(dx) = \frac{c_\alpha}{|x|^{1+\alpha}}dx.$$

Then the characteristic function is given by

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta Z_t}] = \exp(-|\theta|^\alpha t).$$

Introduction

Let $T > 0$ and $X = (X_t)_{0 \le t \le T}$ be a solution of one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t \sigma(X_{s-}) dZ_s, \tag{1}$$

where $t \in [0, T]$, $Z = (Z_t)_{0 \le t \le T}$ is a symmetric α-stable with $\alpha \in (1, 2)$. A generetor of Z is defined by

$$L_\alpha f(x) := \int_{\mathbb{R}\setminus\{0\}} \left\{ f(x+y) - f(x) - 1_{\{|y| \le 1\}} y f'(x) \right\} \frac{dy}{|y|^{1+\alpha}}.$$

We consider uniqueness of the SDE (1).

Introduction

Preceding study 1

Komatsu (1982) proved the pathwise uniqueness of solutions of the SDE (1) under the following conditions. There exists an increasing function $\rho(x)$ on $[0, \infty)$ such that $\rho(0) = 0$, for any $x, y \in \mathbb{R}$,

$$\int_{+0} \rho(x)^{-1} dx = \infty, \quad |\sigma(x) - \sigma(y)|^{\alpha} \le \rho(|x - y|). \tag{2}$$

Ex: $\rho(x) = x \Rightarrow \sigma : (1/\alpha)$-Hölder continuous.

The uniqueness is failed if σ is p-Hölder, $p < 1/\alpha$.

Introduction

We consider this SDE

$$X_t^{(n)} = x_0 + \int_0^t \sigma_n(X_{s-}^{(n)}) dZ_s, . \tag{3}$$

Preceding study 2

Hashimoto (2013) proved that if σ, σ_n are bounded $(\gamma + 1/\alpha)$-Hölder with $\gamma \in [0, 1 - 1/\alpha]$ and $\|\sigma - \sigma_n\|_\infty \leq 1$, then

$$\mathbb{E}[|X_t - X_t^{(n)}|^{\alpha-1}] \leq \begin{cases} C\|\sigma - \sigma_n\|_\infty^{\frac{\alpha^2\gamma}{\alpha\gamma+1}} & \text{if } \gamma \in (0, 1 - 1/\alpha], \\ C\left(\log \dfrac{1}{\|\sigma - \sigma_n\|_\infty}\right)^{-1} & \text{if } \gamma = 0. \end{cases}$$

Ex: $\gamma = 0 \Rightarrow \sigma : 1/\alpha$-Hölder, $\gamma = 1 - 1/\alpha \Rightarrow \sigma$: Lipschitz.

Introduction

Preceding study 3

Hashimoto (2013) proved that if σ, σ_n are bounded $(\gamma + 1/\alpha)$-Hölder with $\gamma \in [0, 1 - 1/\alpha]$ and $\|\sigma - \sigma_n\|_\infty \to 0$, then for any $1 < \beta < \alpha$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t - X_t^{(n)}|^\beta\right] \to 0.$$

Main result 1

Theorem (1)

Consider the following one-dimensional SDEs

$$X_t = x_0 + \int_0^t \sigma(X_{s-})dZ_s, \text{ and } \widetilde{X}_t = \tilde{x}_0 + \int_0^t \tilde{\sigma}(\widetilde{X}_{s-})dZ_s.$$

For any $\beta \in (0, 2(\alpha - 1))$*, suppose that bounded Hölder continuous functions* σ *and* $\tilde{\sigma}$ *satisfy the following conditions:*

$$|\sigma(x) - \sigma(y)| \le K|x-y|^\eta, \ |\tilde{\sigma}(x) - \tilde{\sigma}(y)| \le \widetilde{K}|x-y|^\gamma, \ \eta \in (0,1], \ \gamma \in \left[\frac{\beta+2}{\alpha} - 1, 1\right]$$

$$0 < c_1 < \sigma(x) < c_2, \ \|\tilde{\sigma}\|_\infty < \infty, \ \lambda := \|\sigma - \tilde{\sigma}\|_{L^\alpha(\mathbb{R}, \mu^\alpha_{x_0})} \le 1,$$

where $\mu^\alpha_{x_0}(dy) := (|y - x_0|^{-1-\alpha} \wedge 1)dy$, $\|f\|_{L^\alpha(\mathbb{R}, \mu^\alpha_{x_0})} := \left(\int_{\mathbb{R}} |f(y)|^\alpha \mu^\alpha_{x_0}(dy)\right)^{\frac{1}{\alpha}}$.
Then $\exists C = C(\alpha, \beta, \eta, T, c_1, c_2, K, \widetilde{K}) > 0$,

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t - \widetilde{X}_t|^\beta\right] \le \cdots$$

Main result 1

Theorem (1)

For any $\beta \in (0, 2(\alpha-1))$,

$$|\sigma(x)-\sigma(y)| \le K|x-y|^{\eta},\ |\tilde{\sigma}(x)-\tilde{\sigma}(y)| \le \widetilde{K}|x-y|^{\gamma},\ \eta \in (0,1],\ \gamma \in \left[\frac{\beta+2}{\alpha}-1, 1\right]$$

$$0 < c_1 < \sigma(x) < c_2,\ \|\tilde{\sigma}\|_{\infty} < \infty,\ \lambda := \|\sigma - \tilde{\sigma}\|_{L^{\alpha}(\mathbb{R},\mu^{\alpha}_{x_0})} \le 1,$$

where $\mu^{\alpha}_{x_0}(dy) := (|y-x_0|^{-1-\alpha} \wedge 1)dy$, $\|f\|_{L^{\alpha}(\mathbb{R},\mu^{\alpha}_{x_0})} := \left(\int_{\mathbb{R}} |f(y)|^{\alpha} \mu^{\alpha}_{x_0}(dy)\right)^{\frac{1}{\alpha}}$.
Then, $\exists C = C(\alpha, \beta, \eta, T, c_1, c_2, K, \widetilde{K}) > 0$, $B_{\beta} := 1 \vee 2^{\beta-1}$

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t - \tilde{X}_t\right|^{\beta}\right] \le \begin{cases} B_{\beta}^2|x_0-\tilde{x}_0|^{\beta} + C\lambda^{\frac{\alpha\beta}{2(1+\beta)-\alpha}} & \text{if } \beta \le \frac{\alpha}{2}(\gamma+1)-1, \\ B_{\beta}^2|x_0-\tilde{x}_0|^{\beta} + C\lambda^{\frac{\alpha(\gamma+1)-\beta-2}{\gamma}} & \\ \quad \text{if } \beta > \frac{\alpha}{2}(\gamma+1)-1, & \gamma \in (\frac{\beta+2}{\alpha}-1, 1], \\ B_{\beta}^2|x_0-\tilde{x}_0|^{\beta} + C\left(\log\frac{1}{\lambda}\right)^{-1} & \text{if } \gamma = \frac{\beta+2}{\alpha}-1. \end{cases}$$

Upper bounds of the density

Lemma (1-1, Kulik (2014))

Suppose that $(X_t)_{0\le t\le T}$ is the unique weak solution of this SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)dZ_s, \quad (Z_t)_{0\le t\le T}\text{:symetric } \alpha\text{-stable.}$$

σ^α is γ'-Hölder with $\gamma' \in (0,1]$ and satisfies $0 < b_1 < \sigma^\alpha(x) < b_2$. Then, X_t has a probability density function $p_t(x_0,\cdot)$ for each $t \in (0,T]$. Moreover,

$$p_t(x_0,y) \le C' t^{-\frac{1}{\alpha}}\left(t^{1+\frac{1}{\alpha}} \vee 1\right)\left(|y-x_0|^{-1-\alpha} \wedge 1\right),$$

Proof of Main result 1

We apply a variation of the method introduced by Komatsu.

Lemma (1-2)

For $\varepsilon > 0, \delta > 1$, we can choose a smooth function $\psi_{\delta,\varepsilon}$ which satisfies the following conditions.

$$\psi_{\delta,\varepsilon}(x) = \begin{cases} \textit{between } 0 \textit{ and } 2(x \log \delta)^{-1} & (\varepsilon\delta^{-1} < x < \varepsilon), \\ 0 & (\textit{otherwise}), \end{cases}$$

and $\int_{\varepsilon\delta^{-1}}^{\varepsilon} \psi_{\delta,\varepsilon}(y)dy = 1$. We set $u(x) = |x|^{\beta}$, $u_{\delta,\varepsilon} = u * \psi_{\delta,\varepsilon}$. Then, $u_{\delta,\varepsilon} \in C^2$ and

$$|x|^{\beta} \leq B_{\beta}\varepsilon^{\beta} + B_{\beta}u_{\delta,\varepsilon}(x), \quad u_{\delta,\varepsilon}(x) \leq B_{\beta}|x|^{\beta} + B_{\beta}\varepsilon^{\beta},$$

$$L_{\alpha}u_{\delta,\varepsilon}(\theta) = C_{\alpha,\beta}|\theta|^{\alpha-\beta-1}\psi_{\delta,\varepsilon}(\theta),$$

for any $x \in \mathbb{R}$, $\theta \neq 0$, where $C_{\alpha,\beta}$ is a constant depended on only α and β.

Proof of main result 1

By using the Lévy Itô decomposition

$$Z_t = \int_0^t \int_{|z|<1} z\widetilde{N}(dz, ds) + \int_0^t \int_{|z|\geq 1} zN(dz, ds),$$

where we define the Poisson random measure N as

$$N(F, t) := \sum_{0\leq s\leq t} 1_F(\Delta Z_s),\ F \in \mathcal{B}(\mathbb{R} \setminus \{0\}),\ t \in [0, T].$$

and do the compensated Poisson random measure $\widetilde{N}$ as

$$\widetilde{N}(F, t) := N(F, t) - t\nu(F).$$

Let $f : \mathbb{R} \setminus \{0\} \times [0, T] \times \Omega \to \mathbb{R}$ be a predictable process. If

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \left|f(z, s)\right|^k \nu(dz)ds\right] < \infty, \quad k = 1 \text{ or } 2,$$

then, for each $k = 1,\ 2$, the following process is a L^k martingale.

$$\left(\int_0^t \int_{\mathbb{R}\setminus\{0\}} f(z, s)\widetilde{N}(dz, ds)\right)_{0\leq t\leq T}.$$

Proof of main result 1

We set $Y_t = X_t - \widetilde{X}_t = x_0 - \tilde{x}_0 + \int_0^t (\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s}))dZ_s$.
Using the Itô's formula and $N(dz, ds) = \widetilde{N}(dz, ds) + \nu(dz)ds$,

$$
\begin{aligned}
u_{\delta,\varepsilon}(Y_t) &= u_{\delta,\varepsilon}(Y_0) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left\{ u_{\delta,\varepsilon}(Y_{s-} + (\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s}))z) - u_{\delta,\varepsilon}(Y_{s-}) \right\} \widetilde{N}(dz, ds), \\
&\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big\{ u_{\delta,\varepsilon}(Y_{s-} + (\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s}))z) - u_{\delta,\varepsilon}(Y_{s-}) \\
&\qquad\qquad - 1_{\{|z|\le 1\}}(z)(\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s}))z u'_{\delta,\varepsilon}(Y_{s-}) \Big\} \frac{c_\alpha dz}{|z|^{1+\alpha}} ds \\
&=: u_{\delta,\varepsilon}(Y_0) + M_t^{\delta,\varepsilon} + I_t^{\delta,\varepsilon}, \\
&\le B_\beta |x_0 - \tilde{x}_0|^\beta + B_\beta \varepsilon^\beta + M_t^{\delta,\varepsilon} + I_t^{\delta,\varepsilon}.
\end{aligned}
$$

$(M_t^{\delta,\varepsilon})_{0\le t\le T}$ is a martingale. Using $y = (\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s}))z$ on $I_t^{\delta,\varepsilon}$, as a change of variables

$$
I_t^{\delta,\varepsilon} = c_\alpha \int_0^t \frac{|\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})|^{1+\alpha}}{\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})} L_\alpha u_{\delta,\varepsilon}(Y_s) ds.
$$

Proof of main result 1

By Lemma 1-2,

$$L_\alpha u_{\delta,\varepsilon}(\theta) = C_{\alpha,\beta}|\theta|^{\alpha-\beta-1}\psi_{\delta,\varepsilon}(\theta) \le 2\left|C_{\alpha,\beta}\right| |\theta|^{\alpha-\beta-1}\frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(\theta)}{\theta\log\delta},$$

hence we have

$$I_t^{\delta,\varepsilon} \le 2c_\alpha\left|C_{\alpha,\beta}\right|\int_0^t |\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})|^\alpha |Y_s|^{\alpha-\beta-2}\frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(Y_s)}{\log\delta}ds.$$

Here, since $\tilde{\sigma}$ is γ-Hölder, we obtain

$$|\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})|^\alpha \le 2^{\alpha-1}\widetilde{K}^\alpha|Y_s|^{\alpha\gamma} + 2^{\alpha-1}|\sigma(X_s) - \tilde{\sigma}(X_s)|^\alpha.$$

Therefore, we get

$$\begin{aligned}
|X_t - \widetilde{X}_t|^\beta &\le B_\beta\varepsilon^\beta + B_\beta u_{\delta,\varepsilon}(Y_t)\\
&= B_\beta\varepsilon^\beta + B_\beta u_{\delta,\varepsilon}(Y_0) + B_\beta M_t^{\delta,\varepsilon} + B_\beta I_t^{\delta,\varepsilon}\\
&\le B_\beta^2|x_0 - \tilde{x}_0|^\beta + B_\beta(B_\beta+1)\varepsilon^\beta + B_\beta M_t^{\delta,\varepsilon}\\
&\qquad + \frac{B_\beta\hat{C}_{\alpha,\beta}t\varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log\delta} + \frac{B_\beta\hat{C}_{\alpha,\beta}}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha}\int_0^t|\sigma(X_s) - \tilde{\sigma}(X_s)|^\alpha ds.
\end{aligned}$$

where $\hat{C}_{\alpha,\beta} := 2^\alpha c_\alpha\left|C_{\alpha,\beta}\right|\max\{\tilde{K}^\alpha, 1\}$.

Proof of main result 1

By taking expectation, we obtain

$$\mathbb{E}\left[|X_t - \widetilde{X}_t|^\beta\right] \le B_\beta^2 |x_0 - \tilde{x}_0|^\beta + B_\beta(B_\beta + 1)\varepsilon^\beta + \frac{B_\beta \hat{C}_{\alpha,\beta} t \varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log \delta} + \frac{B_\beta \hat{C}_{\alpha,\beta}}{\log \delta} \left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha} \int_0^t \mathbb{E}\left[|\sigma(X_s) - \tilde{\sigma}(X_s)|^\alpha\right] ds.$$

Here, X_t has a density function $p_t(x_0, \cdot)$, by using Lemma 1-1,

$$\begin{aligned}
\int_0^t \mathbb{E}\left[|\sigma(X_s) - \tilde{\sigma}(X_s)|^\alpha\right] ds &= \int_0^t \int_{\mathbb{R}} \left|\sigma(y) - \tilde{\sigma}(y)\right|^\alpha p_s(x_0, y) dy ds, \\
&\le C' \int_0^t \int_{\mathbb{R}} \left|\sigma(y) - \tilde{\sigma}(y)\right|^\alpha s^{-\frac{1}{\alpha}} \left(s^{1+\frac{1}{\alpha}} \vee 1\right)\left(|y - x_0|^{-1-\alpha} \wedge 1\right) dy ds, \\
&= C' J(t) \|\sigma - \tilde{\sigma}\|^\alpha_{L^\alpha(\mathbb{R}, \mu^\alpha_{x_0})},
\end{aligned}$$

where $J(t) = \int_0^t s^{-\frac{1}{\alpha}} (s^{1+\frac{1}{\alpha}} \vee 1) ds$. We set $\lambda = \|\sigma - \tilde{\sigma}\|_{L^\alpha(\mathbb{R}, \mu^\alpha_{x_0})} \le 1$. Then,

$$\sup_{0 \le t \le T} \mathbb{E}\left[\left|X_t - \tilde{X}_t\right|^\beta\right] \le B_\beta^2 |x_0 - \tilde{x}_0|^\beta + B_\beta(B_\beta + 1)\varepsilon^\beta + \frac{B_\beta \hat{C}_{\alpha,\beta} T \varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log \delta} + \frac{B_\beta \hat{C}_{\alpha,\beta} C' J(T)}{\log \delta} \left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha} \lambda^\alpha.$$

$$\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t-\tilde{X}_t\right|^{\beta}\right]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(\varepsilon^{\beta}+\frac{\varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log\delta}+\frac{1}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha}\lambda^{\alpha}\right)$$

We consider the following three cases.
Case[1]($\beta\le\alpha(\gamma+1)/2-1$): We set $\varepsilon=\lambda^{p}$ $(p>0),\delta=2$. We choose p later. Then, we have

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^{\beta}]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(\lambda^{p\beta}+\lambda^{p(\alpha\gamma+\alpha-\beta-2)}+\lambda^{-p(2+\beta-\alpha)+\alpha}\right).$$

By $p\beta\le p(\alpha\gamma+\alpha-\beta-2)$ and $\lambda\le 1$, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^{\beta}]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(2\lambda^{p\beta}+\lambda^{-p(2+\beta-\alpha)+\alpha}\right).$$

We choose p as $p\beta=-p(2+\beta-\alpha)+\alpha$. Then since $p=\frac{\alpha}{2(1+\beta)-\alpha}$, we get

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t-\widetilde{X}_t|^{\beta}\right]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\lambda^{\frac{\alpha\beta}{2(1+\beta)-\alpha}}.$$

$$\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t-\tilde{X}_t\right|^\beta\right]\le B_\beta^2|x_0-\tilde{x}_0|^\beta+C\left(\varepsilon^\beta+\frac{\varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log\delta}+\frac{1}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha}\lambda^\alpha\right)$$

Case[2]($\beta>\alpha(\gamma+1)/2-1$ and $\gamma\in((\beta+2)/\alpha-1,1]$): We set $\varepsilon=\lambda^p$ $(p>0),\delta=2$. We choose p later. Then, we have

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^\beta]\le B_\beta^2|x_0-\tilde{x}_0|^\beta+C\left(\lambda^{p\beta}+\lambda^{p(\alpha\gamma+\alpha-\beta-2)}+\lambda^{-p(2+\beta-\alpha)+\alpha}\right).$$

By $p\beta>p(\alpha\gamma+\alpha-\beta-2)$ and $\lambda\le 1$, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^\beta]\le B_\beta^2|x_0-\tilde{x}_0|^\beta+C\left(2\lambda^{p(\alpha\gamma+\alpha-\beta-2)}+\lambda^{-p(2+\beta-\alpha)+\alpha}\right).$$

We choose p as $p(\alpha\gamma+\alpha-\beta-2)=-p(2+\beta-\alpha)+\alpha$. Then since $p=1/\gamma$, we get

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t-\widetilde{X}_t|^\beta\right]\le B_\beta^2|x_0-\tilde{x}_0|^\beta+C\lambda^{\frac{\alpha\gamma+\alpha-\beta-2}{\gamma}}.$$

$$\sup_{0\le t\le T}\mathbb{E}\left[\left|X_t-\tilde{X}_t\right|^{\beta}\right]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(\varepsilon^{\beta}+\frac{\varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log\delta}+\frac{1}{\log\delta}\left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha}\lambda^{\alpha}\right)$$

Case[3]($\gamma=(\beta+2)/\alpha-1$): We set $\varepsilon=(\log\frac{1}{\lambda})^{-q},\ \delta=\lambda^{-p}$. Then, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^{\beta}]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left\{\left(\log\frac{1}{\lambda}\right)^{-q\beta}+\frac{1}{p\log\frac{1}{\lambda}}+\frac{\lambda^{\alpha-p(2+\beta-\alpha)}}{p\left(\log\frac{1}{\lambda}\right)^{1-q(2+\beta-\alpha)}}\right\}.$$

We choose q as $-q\beta=-1$, that is $q=1/\beta$.

$$\sup_{t\in[0,T]}\mathbb{E}[|X_t-\widetilde{X}_t|^{\beta}]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(\log\frac{1}{\lambda}\right)^{-1}\left\{1+\frac{1}{p}+\frac{1}{p}\lambda^{\alpha-p(2+\beta-\alpha)}\left(\log\frac{1}{\lambda}\right)^{\frac{2+\beta-\alpha}{\beta}}\right\}.$$

Here, by

$$\sup_{x\in(0,1)}x^{\frac{\alpha}{2}}\left(\log\frac{1}{x}\right)^{\frac{2+\beta-\alpha}{\beta}}<\infty,$$

choosing $p=\alpha/2(2+\beta-\alpha)$, we get

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t-\widetilde{X}_t|^{\beta}\right]\le B_{\beta}^2|x_0-\tilde{x}_0|^{\beta}+C\left(\log\frac{1}{\lambda}\right)^{-1}.$$

This concludes the statement.

Main result 2

Theorem (2)

Consider the following one-dimensional SDEs

$$X_t = x_0 + \int_0^t \sigma(X_{s-})dZ_s, \textit{and } \widetilde{X}_t = \tilde{x}_0 + \int_0^t \tilde{\sigma}(\widetilde{X}_{s-})dZ_s.$$

Suppose that σ and $\tilde{\sigma}$ satisfy the same assumption as Th. (1). Then,
$\exists C = C(\alpha, \beta, \eta, T, c_1, c_2, K, \widetilde{K}) > 0$, $B_\beta := 1 \vee 2^{\beta-1}$

$$\mathbb{P}\left(\sup_{t\in[0,T]} |X_t - \widetilde{X}_t|^\beta > h\right)$$

$$\leq \begin{cases} \frac{B_\beta^2}{h}|x_0 - \tilde{x}_0|^\beta + \frac{C}{h}\lambda^{\frac{\alpha\beta}{2(1+\beta)-\alpha}} & \textit{if } \beta \leq \frac{\alpha}{2}(\gamma+1) - 1, \\ \frac{B_\beta^2}{h}|x_0 - \tilde{x}_0|^\beta + \frac{C}{h}\lambda^{\frac{\alpha(\gamma+1)-\beta-2}{\gamma}} & \textit{if } \beta > \frac{\alpha}{2}(\gamma+1) - 1, \gamma \in (\frac{\beta+2}{\alpha} - 1, 1], \\ \frac{B_\beta^2}{h}|x_0 - \tilde{x}_0|^\beta + \frac{C}{h}\left(\log\frac{1}{\lambda}\right)^{-1} & \textit{if } \gamma = \frac{\beta+2}{\alpha} - 1. \end{cases}$$

Quasi-martingale

$T \in [0, \infty]$, $\Delta t = (t_0, t_1, \ldots, t_{n+1})$ s.t. $0 = t_0 < t_1 < \cdots < t_{n+1} = T$

Definition

The mean variation of X is defined to be

$$V_T(X) := \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{t_i} - X_{t_{i+1}} | \mathcal{F}_{t_i}]\right|\right]$$

Definition

An adapted, càdlàg process X is a ***quasi-martingale*** *on $[0, T]$ if $\mathbb{E}[|X_t|] < \infty$, for each $t \in [0, T]$, and $V_T(X) < \infty$.*

If X is defined on $[0, \infty)$, then we set $X_\infty = 0$.

Property of quasi-martingales

$$V_T(X) := \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{t_i} - X_{t_{i+1}} | \mathcal{F}_{t_i}]\right|\right]$$

Proposition (2-1)

Any martingale, submartingale, and supermartingale X is a quasi-martingale. Furthermore, for $t \geq 0$

$$V_t(X) = \left|\mathbb{E}[X_t - X_0]\right|.$$

Proposition (2-2)

X, Y*: quasi-martingale,* $a \in \mathbb{R}$, $T \in [0, \infty]$

$$\begin{aligned} V_T(a + X) &= V_T(X), \\ V_T(aX) &= |a| V_T(X), \\ V_T(X + Y) &\leq V_T(X) + V_T(Y). \end{aligned}$$

Lemma and theorem for quasimartingale

Theorem (2-3, Rao's theorem (1969))

X is quasimartingale on $[0,\infty)$ $\Leftrightarrow$ $\exists Y, Z$:nonnegative right continuous supermartingale s.t. $X = Y - Z$.

Lemma (2-4, Thomas (1991))

Suppose $T \in [0,\infty), \forall t \geq 0$, $\mathbb{E}[|X_t|], V_t(X) < \infty$, stopping time τ $\Delta\tau = (\tau_0, \tau_1, \ldots, \tau_{n+1})$:stopping times $0 \leq \tau_0 \leq \ldots \leq \tau_{n+1} \leq T$. Then,

$$V_T(X) = \sup_{\Delta\tau} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{\tau_i} - X_{\tau_{i+1}} \mid \mathcal{F}_{\tau_i}]\right|\right],$$

$$\mathbb{E}[|X_{\tau\wedge T}|] \leq V_T(X) + \mathbb{E}[|X_T|] \textit{ for any stopping time } \tau.$$

Lemma (2-5, Thomas (1991))

Suppose $\mathbb{E}[|X_t|] < \infty, \ V_t(X) < \infty$ for $\forall t \in [0, T]$. Then, $\forall c > 0$,

$$c\mathbb{P}(\sup_{0\leq t\leq T} |X_t| > c) \leq V_T(X) + \mathbb{E}[|X_T|].$$

Proof of Rao's theorem

(only if) Let $\Sigma(s)$ denote the set finite subdivision of $[s, \infty]$. We set
$\Delta s_n := (t_0, t_1, \ldots, t_{n+1}) \in \Sigma(s)$, $t_0 = s$, $t_{n+1} = \infty$,
$X^+ := \max(X, 0)$, $X^- := -\min(X, 0)$.

$$Y_s^{\Delta s_n} := \mathbb{E}\left[\sum_{i=0}^{n}\left(\mathbb{E}[X_{t_i} - X_{t_{i+1}}|\mathcal{F}_{t_i}]\right)^+ \middle| \mathcal{F}_s\right], \; Z_s^{\Delta s_n} := \mathbb{E}\left[\sum_{i=0}^{n}\left(\mathbb{E}[X_{t_i} - X_{t_{i+1}}|\mathcal{F}_{t_i}]\right)^- \middle| \mathcal{F}_s\right].$$

Then,

$$Y_s^{\Delta s_n} - Z_s^{\Delta s_n} = X_s.$$

Since $\mathbb{E}[Y_s^{\Delta s_n}] \leq V_\infty(X)$ and $\Sigma(s)$ is a directed set, we define

$$\hat{Y}_s := \lim_{\Delta s_n} Y_s^\sigma, \; \hat{Z}_s := \lim_{\Delta s_n} Z_s^\sigma, \; Y_t := \hat{Y}_{t+}, \; Z_t := \hat{Z}_{t+}. \square$$

Proposition (2-6)

$$V_\infty(X) = \mathbb{E}[Y_0 + Z_0]$$

Lemma and theorem for quasi-martingale

Theorem (2-3, Rao's theorem (1969))

X is quasi-martingale on $[0, \infty) \Leftrightarrow \exists Y, Z$:nonnegative right continuous supermartingale s.t. $X = Y - Z$.

Lemma (2-4, Thomas (1991))

Suppose $T \in [0, \infty), \forall t \geq 0$, $\mathbb{E}[|X_t|], V_t(X) < \infty$, stopping time τ
$\Delta\tau = (\tau_0, \tau_1, \ldots, \tau_{n+1})$:stopping times $0 \leq \tau_0 \leq \ldots \leq \tau_{n+1} \leq T$.
Then,

$$V_T(X) = \sup_{\Delta\tau} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{\tau_i} - X_{\tau_{i+1}} \,\middle|\, \mathcal{F}_{\tau_i}]\right|\right],$$

$$\mathbb{E}[|X_{\tau \wedge T}|] \leq V_T(X) + \mathbb{E}[|X_T|].$$

Lemma (2-5, Thomas (1991))

Suppose $\mathbb{E}[|X_t|] < \infty,\ V_t(X) < \infty$ for $\forall t \in [0, T]$. Then, $\forall c > 0$,

$$c\mathbb{P}(\sup_{0 \leq t \leq T} |X_t| > c) \leq V_T(X) + \mathbb{E}[|X_T|].$$

Proof of Lemma 2-4

We set

$$X_t^T = \begin{cases} X_t - \mathbb{E}[X_T|\mathcal{F}_t] & \text{if } \ t \in [0, T), \\ 0 & \text{if } \ t \geq T. \end{cases}$$

By Rao's theorem, $\exists U^T, \ W^T$:nonnegative supermartingale s.t. $X^T = U^T - W^T$, $V_\infty(X^T) = V_T(X) = \mathbb{E}[U_0^T + W_0^T]$.

$$\begin{aligned} &\mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{\tau_i} - X_{\tau_{i+1}} \mid \mathcal{F}_{\tau_i}]\right|\right], \\ &\leq \mathbb{E}[U_{\tau_0}^T + W_{\tau_0}^T] - \mathbb{E}[U_{\tau_{n+1}}^T + W_{\tau_{n+1}}^T], \\ &\leq \mathbb{E}[U_0^T + W_0^T] = V_T(X). \end{aligned}$$

$$\begin{aligned} \mathbb{E}[|X_{\tau\wedge T}|] &\leq \mathbb{E}\left[\left|\mathbb{E}[X_T - X_{\tau\wedge T}|\mathcal{F}_{\tau\wedge T}]\right|\right] + \mathbb{E}\left[\mathbb{E}[|X_T||\mathcal{F}_{\tau\wedge T}]\right], \\ &\leq V_T(X) + \mathbb{E}[|X_T|]. \square \end{aligned}$$

Lemma and theorem for quasimartingale

Theorem (2-3, Rao's theorem (1969))

X is quasimartingale on $[0, \infty) \Leftrightarrow \exists Y, Z$:nonnegative right continuous supermartingale s.t. $X = Y - Z$.

Lemma (2-4, Thomas (1991))

Suppose $T \in [0, \infty), \forall t \geq 0, \mathbb{E}[|X_t|], V_t(X) < \infty$, stopping time τ $\Delta\tau = (\tau_0, \tau_1, \ldots, \tau_{n+1})$:stopping times $0 \leq \tau_0 \leq \ldots \leq \tau_{n+1} \leq T$. Then,

$$V_T(X) = \sup_{\Delta\tau} \mathbb{E}\left[\sum_{i=0}^{n} \left|\mathbb{E}[X_{\tau_i} - X_{\tau_{i+1}} \mid \mathcal{F}_{\tau_i}]\right|\right],$$

$$\mathbb{E}[|X_{\tau \wedge T}|] \leq V_T(X) + \mathbb{E}[|X_T|].$$

Lemma (2-5, Thomas (1991))

Suppose $\mathbb{E}[|X_t|] < \infty,\ V_t(X) < \infty$ for $\forall t \in [0, T]$. Then, $\forall c > 0$,

$$c\mathbb{P}(\sup_{0 \leq t \leq T} |X_t| > c) \leq V_T(X) + \mathbb{E}[|X_T|].$$

Proof of Lemma 2-5

We set

$$\tau = \inf\{t \in [0, T] : |X_t| > h\}.$$

By Lemma 2-4, we have

$$\begin{aligned} & h\mathbb{P}(\sup_t |X_t| > h), \\ & \leq \mathbb{E}[|X_{\tau \wedge T}|], \\ & \leq V_T(X) + \mathbb{E}[|X_T|]. \square \end{aligned}$$

Proof of main result 2

$u_{\delta,\varepsilon}(Y) = (u_{\delta,\varepsilon}(Y_t))_{0\le t\le T}$ is quasi-martingale on $[0, T]$. In fact,

$$
\begin{aligned}
V_T(u_{\delta,\varepsilon}(Y)) &= V_T(u_{\delta,\varepsilon}(Y_0) + M^{\delta,\varepsilon} + I^{\delta,\varepsilon}) \\
&= \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{m} \left|\mathbb{E}[(M^{\delta,\varepsilon}_{t_i} + I^{\delta,\varepsilon}_{t_i}) - (M^{\delta,\varepsilon}_{t_{i+1}} + I^{\delta,\varepsilon}_{t_{i+1}})|\mathcal{F}_{t_i}]\right|\right], \\
&\le \sup_{\Delta t} \mathbb{E}\left[\sum_{i=0}^{m} \mathbb{E}\left[\left|I^{\delta,\varepsilon}_{t_i} - I^{\delta,\varepsilon}_{t_{i+1}}\right| \Big|\mathcal{F}_{t_i}\right]\right], \\
&= \sup_{\Delta t} \sum_{i=0}^{m} \mathbb{E}\left[\left|\int_{t_i}^{t_{i+1}} \frac{|\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})|^{1+\alpha}}{\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})} L_\alpha u_{\delta,\varepsilon}(Y_s) ds\right|\right], \\
&\le 2c_\alpha \left|C_{\alpha,\beta}\right| \mathbb{E}\left[\int_0^T |\sigma(X_s) - \tilde{\sigma}(\widetilde{X_s})|^{\alpha} |Y_s|^{\alpha-\beta-1} \frac{1_{[\varepsilon\delta^{-1},\varepsilon]}(Y_s)}{Y_s \log \delta} ds\right], \\
&\le \frac{\hat{C}_{\alpha,\beta} T \varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log \delta} + \frac{\hat{C}_{\alpha,\beta} C' J(T)}{\log \delta} \left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha} \lambda^\alpha, \\
&< \infty.
\end{aligned}
$$

Proof of main result 2

By Lemma 1-2,

$$|Y_t|^\beta \leq B_\beta \varepsilon^\beta + B_\beta u_{\delta,\varepsilon}(Y_t),$$

We get by using Lemma 2-5

$$\begin{aligned}
h\mathbb{P}\left(\sup_{t\in[0,T]} |Y_t|^\beta > h\right) &\leq h\mathbb{P}\left(\sup_{t\in[0,T]} (B_\beta \varepsilon^\beta + u_{\delta,\varepsilon}(Y_t)) > h\right), \\
&\leq V_T(B_\beta \varepsilon^\beta + u_{\delta,\varepsilon}(Y)) + \mathbb{E}[|B_\beta \varepsilon^\beta + B_\beta u_{\delta,\varepsilon}(Y_T)|], \\
&\leq V_T(u_{\delta,\varepsilon}(Y)) + \mathbb{E}[B_\beta \varepsilon^\beta + B_\beta u_{\delta,\varepsilon}(Y_T)], \\
&\leq B_\beta^2 |x_0 - \tilde{x}_0|^\beta + B_\beta \left(B_\beta + 1\right) \varepsilon^\beta \\
&\quad + \frac{(1 + B_\beta)\hat{C}_{\alpha,\beta} T \varepsilon^{\alpha\gamma+\alpha-\beta-2}}{\log \delta} \\
&\quad + \frac{(1 + B_\beta)\hat{C}_{\alpha,\beta} C' J(T)}{\log \delta} \left(\frac{\delta}{\varepsilon}\right)^{2+\beta-\alpha} \lambda^\alpha.
\end{aligned}$$

The remaining proof is the same as main result 1.□

Future issues

- d-dimension
- Estimation for $\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t - \widetilde{X}_t\right|^\beta\right]$
- SDE $X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s-})dZ_s$

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Thank you for your attention